LASER PHYSICS

Fabien Bretenaker

2015

1. Laser and matter-light interaction
2. Equations of the single-frequency laser
3. Single-frequency laser in steady-state regime
4. Laser based on an inhomogeneously broadened transition
5. Transient and Q-switched operations
6. Frequency and intensity noises
7. Two-frequency lasers
8. Mode-locked laser operation
9. Propagation and characterization of short laser pulses
10. Open resonators I: Modes and rays
11. Open resonators II: Gaussian beams
12. Open resonators III: Stable cavity modes
13. Open resonators IV: Unstable cavities
Laser Physics

Fabien Bretenaker
Fabien.Bretenaker@u-psud.fr

Laboratoire Aimé Cotton
CNRS - Université Paris Sud, ENS Cachan, Orsay, France
&
Département de Physique - Ecole Polytechnique, Palaiseau, France

March 30, 2015
Foreword

The aim of this lecture is to give an introduction to the physical principles underlying laser operation. Starting only from basic quantum mechanics and electromagnetism, we derive the characteristics of the laser gain and the laser dynamical equations. This permits us to deduce the behavior of single-mode lasers in steady-state, transient, and pulsed regime. We also introduce the basic concepts related to laser noise. We then generalize this physics to multimode lasers dynamics, eventually allowing us to describe the mode-locked laser operation. We end up by describing the transverse characteristics of the laser intracavity electromagnetic field, illustrating the concept of mode in both stable and unstable cavities.

To build this lecture, we have benefited from the invaluable experience of Christian Delsart who taught laser physics in Orsay during many years. We have also been much influenced by the way Alain Aspect teaches laser physics (or “physics using lasers”) in Ecole Polytechnique. We have also benefited from the courses given by Marie-Claire Schanne-Klein and Antonello de Martino and been influenced by the reading of Nicolas Forget’s thesis. A large part of these lecture notes has been integrally plagiarized from these sources, and also from other sources like the books mentioned in the bibliography section below (most particularly Tony Siegman’s book entitled Lasers). We are much indebted to these friends. We also wish to thank Cyril Drag, followed by Ghaya Baili, Marc Hanna, Alexios Beveratos, and Frédéric Druon for having accepted to take part in this course, and also for their help in the preparation of this course and for the enlightening discussion we have and the friendly atmosphere in which this teaching is performed. Finally, we wish to thank Nicolas Dubreuil and Jacques Robert for their encouragements and for having trusted us.

This lecture is deliberately focused on basic laser physics. It does not deal with laser technology or laser applications, even if these are very exciting fields. We are also aware of the fact that important subjects are not covered by this lecture, such as, e.g., semiconductor lasers. This results from a deliberate choice taking into account the limited volume of this course and
the fact that this is the first introduction to lasers for some of the students attending this course.

Fabien Bretenaker
Orsay, July 2012
Bibliography

There are many books dealing with laser physics. Here is an incomplete list:


Index of symbols

\( a \): i) amplitude noise; ii) aperture half-size in an unstable cavity.
\( A \): spontaneous emission decay rate.
\( \mathcal{A} \): complex amplitude of a plane wave.
\( \mathcal{A}_1 \): complex amplitude of the master laser.
\( \mathcal{A}_{1\text{phot}} \): amplitude of the intracavity field for one photon in the mode.
\( c_0 \): light velocity in vacuum.
\( C \): i) coupling constant; ii) chirp parameter.
\( d_{12}, d \): electric dipole matrix element.
\( \mathbf{d} \): electric dipole vector operator.
\( d\mathbf{T}_i \): infinitesimal transfer operator.
\( D \): group delay dispersion.
\( D_a \): diffusion coefficient for the amplitude.
\( D_\varphi \): diffusion coefficient for the phase.
\( \mathcal{E} \): pulse envelope.
\( \mathbf{E} \): “real” vectorial electric field.
\( E_0 \): pulse amplitude.
\( \tilde{E}_0 \): pulse Fourier transform amplitude.
\( E^{(+)} \): analytic signal.
\( \tilde{E}^{(+)} \): Fourier transform of the analytic signal.
\( f \): focal length.
\( f_{\text{anti}} \): antiphase frequency.
\( f_{\text{beat}} \): beatnote frequency.
\( f_{\text{relax}} \): relaxation oscillation frequency.
\( F \): number of photons in the laser mode.
INDEX OF SYMBOLS

\( F_{\text{sat}} \): saturation photon number.
\( g \): normalized line profile.
\( g_1, g_2 \): two-mirror cavity \( g \) parameters.
\( G \): Gaussian profile.
\( G_0 \): intensity gain.
\( G_D \): Doppler profile.
\( h \): Planck’s constant.
\( \hbar \): reduced Planck’s constant.
\( \hat{H} \): atomic Hamiltonian.
\( \hat{H}_I \): interaction Hamiltonian.
\( H_n \): Hermite polynomial of order \( n \).
\( I \): light intensity.
\( I_{\text{out}} \): output intensity.
\( I_{\text{sat}} \): saturation intensity.
\( I_{\text{sat}0} \): saturation intensity at line center.
\( I_{\text{sat,abs}} \): absorber saturation intensity.
\( k \): light wavenumber.
\( k_B \): Boltzmann’s constant.
\( k_P \): wavenumber at the carrier frequency.
\( K \): Huygens-Fresnel kernel.
\( L \): i) Lorentzian. ii) Two-mirror cavity length.
\( L_0 \): optical thickness of a paraxial system.
\( L_a \): length of the active medium.
\( L_{\text{cav}} \): length of one cavity round-trip.
\( L_{\text{cav,opt}} \): optical length of one cavity round-trip.
\( L_p^n \): generalized Laguerre polynomial of orders \( n \) and \( p \).
\( m \): half-trace of the \( ABCD \) matrix.
\( M \): transverse magnification.
\( n \): atomic density.
\( n_0 \): refractive index.
\( n_2 \): Kerr effect coefficient.
$N$: total number of atoms.
$N_c$: collimated Fresnel number.
$N_{eq}$: equivalent Fresnel number.
$N_f$: resonator Fresnel number.
$N_i$: population of level $i$ (in terms of number of atoms).
$P$: i) polarization; ii) Inhomogeneous profile.
$\mathcal{P}$: complex amplitude of the polarization.
$q$: complex radius of curvature.
$\hat{q}$: reduced complex radius of curvature.
$q_a, q_b$: eigensolutions of an unstable cavity.
$Q_a$: active medium quality factor.
$Q_{cav}$: cavity quality factor.
$r$: i) relative excitation; ii) ray position.
$r'$: reduced slope of the ray.
$r$: ray vector.
$r_+, r_-$: eigenrays.
$R$: i) ideal gas constant; ii) radius of curvature.
$\hat{R}$: reduced radius of curvature.
$R_a, R_b$: radii of curvature of the eigensolutions of an unstable cavity.
$R_e$: effective radius of curvature.
$R_i$: intensity reflection coefficient of mirror $i$.
$S$: mode area.
$S_a$: power spectral density of amplitude fluctuations.
$S_{E(+)}$: laser field spectrum.
$S_{\delta F}$: power spectral density of the photon number noise.
$S_{\delta \nu}$: power spectral density of the frequency noise.
$S_{\zeta}$: power spectral density of $\zeta$.
$S_{\zeta_1, \zeta_2}$: correlation spectrum of $\zeta_1$ and $\zeta_2$.
t_a$: field transmission of the active medium.
$T$: i) transmission of the output coupler; ii) Gas temperature.
$T_a$: intensity transmission of the active medium.
$T_{\text{build-up}}$: pulse build-up time.
$T_g$: group delay.
$T_i$: transfer operator.
$u$: energy density of a traveling wave.
$\mathbf{u}$: polarization and intracavity spatial field distribution.
$\mathbf{U}$: transverse field structure.
$V$: spatial part of the electric field.
$u_{\text{sat}}$: saturation energy density.
$u_{\text{standing}}$: energy density of a standing wave.
$\mathbf{U}_{mn}$: cavity eigenmode.
$v_g$: group velocity.
$V_a$: mode volume in the active medium.
$V_{\text{cav}}$: mode volume in the cavity.
$w$: Gaussian beam radius.
$w_0$: Gaussian beam radius at the waist.
$W_P$: pumping probability per unit time.
$\mathcal{W}$: Electromagnetic energy stored inside the cavity.
$\mathcal{W}_1, \mathcal{W}_2$: collimated field distributions in an unstable cavity.
$z_R$: Rayleigh range.
$\alpha$: gain coefficient.
$\alpha_0$: unsaturated gain coefficient.
$\alpha_{\text{abs}}$: absorption coefficient.
$\alpha_{\text{th}}$: gain coefficient at threshold.
$\beta$: damping coefficient for the amplitude fluctuations.
$\beta_2$: group delay dispersion.
$\gamma_{\text{geom}}$: geometric eigenvalue of an unstable cavity.
$\gamma_i$: decay rate of the population of level $i$.
$\gamma_{mn}$: transverse mode eigenvalue.
$\Gamma$: decay rate of the coherences.
$\Gamma_x$: auto-correlation of the process $x$.
$\Gamma_{sp}$: rate of spontaneous emission into the laser mode.
δ: detuning.
δ_{cav}: detuning with respect to the cavity resonance frequency.
δ_{K}: Kerr effect parameter.
δF: photon number noise.
δN: population inversion noise.
δT: total time delay.
δT_{M}: modulator time delay.
δT_{gv}: group delay.
δν: frequency noise.
δϕ: phase excursion.
δω_{a}: active medium bandwidth.
Δ: free spectral range.
Δ_{L}: transverse Laplacian.
Δn: population inversion (in terms of number of atoms per unit volume).
Δn_{0}: pumping rate (in terms of number of atoms per unit volume).
Δn_{th}: population inversion at threshold (in terms of number of atoms per unit volume).
ΔN: population inversion (in terms of number of atoms).
ΔN_{0}: pumping rate (in terms of number of atoms).
ΔN_{i}: population inversion at the beginning of a pulse.
ΔN_{th}: population inversion at threshold (in terms of number of atoms).
Δt: pulse duration.
Δt_{0}: steady-state pulse duration.
Δu: generalized complex spectral width.
Δν: linewidth.
Δν_{D}: Doppler linewidth.
Δω: pulse spectrum width.
ε: relative permittivity.
ε_{0}: vacuum permittivity.
ε_{abs}: saturable absorber coefficient.
\( \zeta \): characteristic quantity for the pulse build-up.
\( \zeta_A \): Langevin force for the field amplitude.
\( \zeta_F \): Langevin force for the number of photons.
\( \zeta_{\Delta N} \): Langevin force for the population inversion.
\( \eta \): i) extra cavity losses; ii) Q-switched laser yield.
\( \theta \): i) field transmission of the modulator; ii) obliquity angle.
\( \theta_0 \): maximum field transmission of the modulator.
\( \theta_i \): angle of incidence in medium \( i \).
\( \theta_{1/e} \): beam divergence angle.
\( \Theta \): intensity transmission of the modulator.
\( \kappa \): atom-field coupling coefficient.
\( \lambda_a, \lambda_b \): eigenvalues of the \( ABCD \) matrix of an unstable cavity.
\( \Lambda_i \): pumping rate of level \( i \).
\( \mu \): modulation depth.
\( \mu_0 \): vacuum permeability.
\( \nu \): light frequency.
\( \nu_0 \): Bohr frequency of the transition.
\( \nu_q \): cavity resonance frequency.
\( \xi_{12}, \xi_{21}, \xi \): ratios of the cross- to self-saturation coefficients.
\( \varpi \): chirp rate.
\( \Pi \): losses per round-trip.
\( \Pi_{\text{geom}} \): geometric losses of an unstable cavity.
\( \rho \): eikonal.
\( \hat{\rho} \): density operator.
\( \sigma \): laser cross section.
\( \sigma_{12}, \sigma_{21} \): atomic coherences in the rotating frame.
\( \sigma_a \): variance of the amplitude fluctuations.
\( \varsigma \): fictitious conductivity in the cavity.
\( \tau \): population inversion lifetime.
\( \tau_{\text{cav}} \): cavity photon lifetime.
\( \tau_{\text{damp}} \): relaxation oscillation damping time.
\( \tau_{\text{inj}} \): injection rate of the master laser field.
\( \tau_{\text{relax}} \): laser relaxation time.
\( \Upsilon \): cavity losses other than the transmission of the output coupler.
\( \varphi \): phase of the field.
\( \varphi_1 \): phase of the master laser field.
\( \Phi \): photon flux.
\( \psi \): Gouy phase shift.
\( \chi \): susceptibility.
\( \chi_{\text{at}} \): susceptibility of the active atoms.
\( \chi_{\text{mat}} \): susceptibility of the matrix.
\( \omega \): angular frequency of light.
\( \omega_0 \): Bohr angular frequency of the transition.
\( \omega_1 \): i) complex Rabi angular frequency; ii) master laser angular frequency
\( \omega_L \): lock-in threshold.
\( \omega_M \): modulator angular frequency.
\( \omega_{\text{mp}} \): angular eigenfrequency of a given transverse and longitudinal mode.
\( \omega_p \): pulse carrier angular frequency.
\( \omega_q \): cavity resonance angular frequency.
\( \Omega \): nutation angular frequency.
\( \Omega_{\text{relax}} \): relaxation oscillation angular frequency.
## Contents

- Foreword iii
- Bibliography v
- Index of symbols vii

### 1 Laser and matter-light interaction 1
  - 1.1 Introduction and history 1
  - 1.1.1 History 1
  - 1.1.2 What is a laser? 3
  - 1.1.3 Outlook of the first part of this lecture 4
  - 1.2 The LASER process 4
  - 1.2.1 Useful energetic quantities 4
  - 1.2.2 Energy conversion processes 5
  - 1.2.3 Heuristic derivation of the equations of the single-frequency laser 7
  - 1.2.4 Steady-state regime: saturation and threshold 9
  - 1.3 Optical Bloch equations for a two-level system 11
  - 1.3.1 Electric-dipole interaction. Rabi frequency 11
  - 1.3.2 Density Matrix (or Density Operator) 13
  - 1.3.3 Relaxation. Pumping 14
  - 1.4 Steady-state regime. Gain. Saturation 15
  - 1.4.1 Steady-state regime 15
  - 1.4.2 Susceptibility. Saturation 15
  - 1.4.3 Gain. Dispersion 17
  - 1.5 Laser cross section. Generalizations 18
  - 1.5.1 Definition of the cross section 18
  - 1.5.2 Generalization to degenerate levels 19
  - 1.5.3 Generalization to more complicated profiles 19
2 Equations of the single-frequency laser 21
2.1 Bloch-Maxwell equations of the laser 21
  2.1.1 Equation of evolution for the polarization 22
  2.1.2 Equation of evolution of the population inversion 22
  2.1.3 Equation of evolution of the field 23
  2.1.4 Cavity losses and detuning 24
2.2 Adiabatic elimination. Laser classes 25
  2.2.1 Discussion. Laser dynamic variables 25
  2.2.2 Lasers dynamic classes 26
  2.2.3 Adiabatic elimination of the polarization 27
  2.2.4 Equations of evolution in terms of numbers of photons
     and atoms 28
  2.2.5 Case where the active medium does not fill the cavity 29
2.3 Rate equations. Three- and four-level systems 30
  2.3.1 Rate equation approximation: general case 30
  2.3.2 Three-level system 30
  2.3.3 Four-level system 32
  2.3.4 Standard equations 34
  2.3.5 Introduction of spontaneous emission 34
  2.3.6 Pumping mechanisms 35

3 Single-frequency laser in steady-state regime 37
3.1 Steady-state solutions 37
  3.1.1 Determination of the steady-state solutions for $F$ and
     $\Delta N$ 37
  3.1.2 Stability of the steady-state solutions 38
  3.1.3 Summary. Comparison between three- and four-level
     systems 41
3.2 Laser frequency 41
  3.2.1 Cavity modes. Frequency Pulling 41
  3.2.2 Single-frequency operation 45
3.3 Laser power 47
  3.3.1 Optimal output coupling 47
  3.3.2 Power in the vicinity of threshold 48
3.4 Spatial hole burning in a linear cavity 50
  3.4.1 Standing wave. Saturation 50
  3.4.2 Output power 51
  3.4.3 Multimode operation 52
CONTENTS

4 Laser based on an inhomogeneously broadened medium 55
  4.1 Inhomogeneously broadened medium ............................. 55
    4.1.1 Gaussian profile. Doppler effect. Ions embedded in a
          crystalline matrix ........................................ 55
    4.1.2 Unsaturated amplification coefficient .................... 57
    4.1.3 Spectral hole burning .................................... 58
    4.1.4 Saturated amplification coefficient ....................... 60
  4.2 Laser operation ............................................ 60
    4.2.1 Application of the oscillation condition. Multimode
          operation ................................................. 60
    4.2.2 Output power of a given mode ........................... 62
    4.2.3 Mode competition: beyond the “rate equations” model .... 62
  4.3 Special case of gas lasers ..................................... 62
    4.3.1 Spectral hole burning in a linear cavity ................ 62
    4.3.2 Lamb dip ................................................... 63
    4.3.3 Reverse Lamb dip ......................................... 64
    4.3.4 Spatial hole burning ..................................... 64
  4.4 Mode selection ................................................ 65
    4.4.1 Different techniques to make the laser monomode ... 65
    4.4.2 Laser frequency stabilization ............................ 66

5 Transient and Q-switched operations 67
  5.1 Transient laser behavior ....................................... 67
    5.1.1 Description in single-frequency regime: relaxation os-
          cillations ............................................... 67
    5.1.2 Switching on the laser: spiking .......................... 70
  5.2 Transient operation in multimode regime ....................... 72
  5.3 Q-switched laser ............................................... 73
    5.3.1 Principe .................................................. 73
    5.3.2 Q-switching techniques ................................. 74
    5.3.3 Theory of active Q-switching ............................ 76
    5.3.4 Example .................................................. 79

6 Frequency and intensity noises 81
  6.1 Langevin equation for the class-A laser ...................... 81
    6.1.1 Equation of evolution of the intensity ................ 81
    6.1.2 Equation of evolution for the field ...................... 82
    6.1.3 Heuristic introduction of spontaneous emission ........ 84
  6.2 Amplitude and phase noises of a class-A laser ............... 86
    6.2.1 Amplitude noise ......................................... 87
    6.2.2 Phase noise .............................................. 88
<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.2.3 Laser linewidth</td>
<td>89</td>
</tr>
<tr>
<td>6.3 General case and application to the class-B laser</td>
<td>91</td>
</tr>
<tr>
<td>6.3.1 Derivation of the Langevin equations</td>
<td>92</td>
</tr>
<tr>
<td>6.3.2 Application to the calculation of the RIN of class-A and class-B lasers</td>
<td>93</td>
</tr>
<tr>
<td>7 Two-frequency lasers</td>
<td>97</td>
</tr>
<tr>
<td>7.1 Self- and cross-saturation terms</td>
<td>98</td>
</tr>
<tr>
<td>7.2 Mode competition in class-A lasers</td>
<td>99</td>
</tr>
<tr>
<td>7.2.1 Simultaneous oscillation of the two modes</td>
<td>100</td>
</tr>
<tr>
<td>7.2.2 Oscillation of one mode only</td>
<td>102</td>
</tr>
<tr>
<td>7.2.3 Bistability</td>
<td>103</td>
</tr>
<tr>
<td>7.3 Transient behavior of a two-frequency class-B laser</td>
<td>103</td>
</tr>
<tr>
<td>7.3.1 Standard relaxation oscillations</td>
<td>104</td>
</tr>
<tr>
<td>7.3.2 Antiphase relaxation oscillations</td>
<td>105</td>
</tr>
<tr>
<td>7.4 Injection locking</td>
<td>106</td>
</tr>
<tr>
<td>7.4.1 Equations of the injected laser</td>
<td>106</td>
</tr>
<tr>
<td>7.4.2 Behavior in the case of a weak injection</td>
<td>108</td>
</tr>
<tr>
<td>8 Mode-locked laser operation</td>
<td>113</td>
</tr>
<tr>
<td>8.1 Introduction</td>
<td>113</td>
</tr>
<tr>
<td>8.2 Spectral approach to active mode-locking</td>
<td>116</td>
</tr>
<tr>
<td>8.3 Temporal approach to active mode-locking I: Kuizenga and Siegman’s model</td>
<td>120</td>
</tr>
<tr>
<td>8.3.1 Passage through the amplifier</td>
<td>121</td>
</tr>
<tr>
<td>8.3.2 Passage through the modulator</td>
<td>122</td>
</tr>
<tr>
<td>8.3.3 Evolution after one cavity round-trip</td>
<td>123</td>
</tr>
<tr>
<td>8.3.4 Steady-state regime</td>
<td>123</td>
</tr>
<tr>
<td>8.3.5 Build-up time of the pulsed regime</td>
<td>124</td>
</tr>
<tr>
<td>8.3.6 Need for a more elaborate model</td>
<td>126</td>
</tr>
<tr>
<td>8.4 Temporal approach to active mode-locking II: Haus’s model</td>
<td>127</td>
</tr>
<tr>
<td>8.4.1 Laser amplifier</td>
<td>128</td>
</tr>
<tr>
<td>8.4.2 Modulator</td>
<td>128</td>
</tr>
<tr>
<td>8.4.3 Intracavity dispersion</td>
<td>129</td>
</tr>
<tr>
<td>8.4.4 Intracavity losses</td>
<td>130</td>
</tr>
<tr>
<td>8.4.5 Master equation</td>
<td>130</td>
</tr>
<tr>
<td>8.4.6 Solution without dispersion ($D_\omega = 0$)</td>
<td>131</td>
</tr>
<tr>
<td>8.4.7 Solution in the presence of dispersion ($D_\omega \neq 0$)</td>
<td>132</td>
</tr>
<tr>
<td>8.5 Temporal approach to passive mode-locking</td>
<td>133</td>
</tr>
<tr>
<td>8.5.1 Fast saturable absorber</td>
<td>134</td>
</tr>
<tr>
<td>8.5.2 Kerr effect</td>
<td>134</td>
</tr>
</tbody>
</table>
## CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.5.3 Master equation</td>
<td>134</td>
</tr>
<tr>
<td>9 Propagation and characterization of short laser pulses</td>
<td>137</td>
</tr>
<tr>
<td>9.1 Dispersion. Chirp</td>
<td>137</td>
</tr>
<tr>
<td>9.1.1 Time-frequency Fourier transform: some useful definitions and relations</td>
<td>137</td>
</tr>
<tr>
<td>9.1.2 Propagation of a Gaussian pulse in a dispersive medium</td>
<td>138</td>
</tr>
<tr>
<td>9.1.3 Propagation of a chirped pulse in a dispersive medium</td>
<td>142</td>
</tr>
<tr>
<td>9.2 Pulse characterization</td>
<td>144</td>
</tr>
<tr>
<td>9.2.1 Intensity autocorrelation</td>
<td>144</td>
</tr>
<tr>
<td>9.2.2 Interferometric autocorrelation</td>
<td>145</td>
</tr>
<tr>
<td>9.2.3 Frequency-Resolved Optical Gating (FROG)</td>
<td>148</td>
</tr>
<tr>
<td>10 Optical resonators I: Modes and rays</td>
<td>153</td>
</tr>
<tr>
<td>10.1 Introduction: the concept of mode</td>
<td>153</td>
</tr>
<tr>
<td>10.1.1 Propagation kernel</td>
<td>153</td>
</tr>
<tr>
<td>10.1.2 Eigenmode</td>
<td>154</td>
</tr>
<tr>
<td>10.1.3 The Fox and Li approach</td>
<td>155</td>
</tr>
<tr>
<td>10.1.4 An analytical approach</td>
<td>156</td>
</tr>
<tr>
<td>10.2 Ray optics and ray matrices</td>
<td>156</td>
</tr>
<tr>
<td>10.2.1 Vector description of optical rays</td>
<td>156</td>
</tr>
<tr>
<td>10.2.2 Ray matrices</td>
<td>157</td>
</tr>
<tr>
<td>10.2.3 Cascaded $ABCD$ systems</td>
<td>161</td>
</tr>
<tr>
<td>10.2.4 Transmission of a spherical wave through an $ABCD$ system</td>
<td>161</td>
</tr>
<tr>
<td>10.2.5 Evolution of rays in a periodic system</td>
<td>162</td>
</tr>
<tr>
<td>11 Optical resonators II: Gaussian beams</td>
<td>167</td>
</tr>
<tr>
<td>11.1 Huygens integral</td>
<td>167</td>
</tr>
<tr>
<td>11.1.1 Paraxial wave equation</td>
<td>167</td>
</tr>
<tr>
<td>11.1.2 Huygens integral in the Fresnel approximation</td>
<td>168</td>
</tr>
<tr>
<td>11.2 Gaussian beams</td>
<td>170</td>
</tr>
<tr>
<td>11.2.1 Complex spherical wave</td>
<td>170</td>
</tr>
<tr>
<td>11.2.2 Higher-order Gaussian modes</td>
<td>171</td>
</tr>
<tr>
<td>11.3 Physical properties of Gaussian beams</td>
<td>173</td>
</tr>
<tr>
<td>11.3.1 Fundamental mode</td>
<td>173</td>
</tr>
<tr>
<td>11.3.2 Hermite-Gaussian modes</td>
<td>175</td>
</tr>
<tr>
<td>11.3.3 Laguerre-Gaussian modes</td>
<td>175</td>
</tr>
</tbody>
</table>
12 Optical resonators III: Stable cavity modes \hspace{1cm} 179
12.1 Stable two-mirror cavities \hspace{1cm} 179
12.1.1 Derivation of the mode \hspace{1cm} 179
12.1.2 Stability diagram \hspace{1cm} 181
12.1.3 Frequencies of the transverse modes \hspace{1cm} 182
12.2 $ABCD$ matrices and Gaussian beams \hspace{1cm} 183
12.2.1 Generalization of the Huygens integral to any paraxial
    system \hspace{1cm} 185
12.2.2 The $ABCD$ formalism for Gaussian beams \hspace{1cm} 188
12.2.3 Application to cavities \hspace{1cm} 189
12.3 Mode losses \hspace{1cm} 190

13 Optical resonators IV: Unstable cavities \hspace{1cm} 193
13.1 Geometrical analysis of unstable resonators \hspace{1cm} 193
13.1.1 Unstable resonator eigenwaves \hspace{1cm} 194
13.1.2 Positive branch unstable resonators \hspace{1cm} 194
13.1.3 Negative branch unstable resonators \hspace{1cm} 195
13.1.4 Real unstable resonators \hspace{1cm} 195
13.1.5 Geometrical output coupling value \hspace{1cm} 196
13.2 Role of diffraction in unstable resonators \hspace{1cm} 198
13.2.1 Canonical formulation \hspace{1cm} 198
13.2.2 Collimated Fresnel number \hspace{1cm} 200
13.2.3 The demagnifying solution \hspace{1cm} 200
13.2.4 Equivalent Fresnel number \hspace{1cm} 201
13.2.5 Mode losses \hspace{1cm} 202
Chapter 1

Laser and matter-light interaction

1.1 Introduction and history

1.1.1 History

The acronym “M.A.S.E.R.” (Microwave Amplification by Stimulated Emission of Radiation) was introduced in 1955 (J. P. Gordon et al., Phys. Rev. 99 (1955) 1264). The acronym “L.A.S.E.R.” (Light Amplification by Stimulated Emission of Radiation) was first introduced in 1964 to replace the name “Optical Maser”. This is a short chronology of the history of the LASER:

- 1917 : A. Einstein, stimulated emission, explanation of Planck’s law.
- 1960 : T. H. Maiman, pulsed ruby laser ($\lambda = 694.3$ nm).
- End 1960 : A. Javan et al., continuous wave (cw) helium-neon laser ($\lambda = 1.15$ $\mu$m).
- 1961 : A. Javan et al., cw helium-neon laser ($\lambda = 633$ nm).
1. LIGHT-MATTER INTERACTION

- 1961: P. A. Franken et al., frequency doubling of a ruby laser using a quartz crystal.
- 1963: P. Kumar et al., CO$_2$ laser.
- 1964: Ar$^+$ and Nd$^{3+}$:YAG lasers.
- 1966: Pulsed dye laser (red, orange, yellow).
- 1975: Cw semiconductor laser.
- 1977: D. Deacon et al., first free electron laser.
- 1993: S. Nakamura et al., blue diode laser based on GaN.
- 1997: Visible petawatt pulsed laser ($10^{15}$ W).


Since 1961, many new lasers have been developed every year. Nowadays, most laser developments deal with “all solid-state” lasers (semiconductor lasers, lasers based on rare earth doped crystalline or amorphous matrices, fiber lasers...).
1.1.2 What is a laser?

Lasers are devices that produce coherent radiation with a wavelength lying in the infrared (IR), visible, or ultraviolet (UV) part of the electromagnetic spectrum. Masers are based on the same principles but emit radiation in the microwave domain. This lecture focuses on lasers only: these devices may use an amazing variety of amplifying materials and of amplification mechanisms, and have led or will lead to numerous applications.

![Schematics of the laser principle.](image)

Figure 1.1: Schematics of the laser principle.

In spite of such a diversity, we can define the four main elements of a laser according to Figure 1.1:

1. An active medium based on atoms, molecules, ions, or electrons in a gas, a plasma, a liquid or a solid-state medium. Its role is to amplify a wave traveling through it.

2. An excitation scheme (also called “pumping”) which allows to turn the active medium into an amplifier for the electromagnetic radiation (example: “population inversion” obtained by optical pumping).

3. An optical resonator (or “cavity”) which allows to turn the system into a resonant oscillator (linear Fabry-Perot cavities, ring cavities, etc.).

4. An output coupler which allows to use part of the radiation stored inside the cavity (output coupling mirror, beam splitter, etc.).

Many other devices can also be implemented inside or outside the resonator, such as, e.g., dispersive elements aiming at shaping the spectral
properties of the laser to meet the user’s needs, commutators that create pulses, nonlinear media which convert the optical frequency (second harmonic generation, third harmonic generation, frequency sum, frequency difference, Raman scattering, parametric conversion, etc...).

1.1.3 Outlook of the first part of this lecture

Chapter 1 describes the different interaction processes between the laser light and the atoms of the laser active medium. Starting from first principles (Maxwell equations and Schrödinger’s equation), we derive in chapter 2 the Bloch-Maxwell equations of the monomode laser. Chapter 3 solves these equations in steady-state regime. Chapter 4 deals with inhomogeneously broadened media. Chapter 5 describes the behavior of lasers in transient and Q-switched regimes. Finally, chapter 6 gives a semi-classical description of laser noise.

The behavior of multimode lasers, including in particular the mode-locked operation regime, and the physics of laser beams and laser cavities will be described in the second part of these lecture notes.

1.2 The LASER process

1.2.1 Useful energetic quantities

The definition of the following quantities is worth knowing, together with their units in the international system. We also give their French translation:


Fluence, “densité d’énergie surfacique” $E_{\text{surf}}$ [J.m$^{-2}$] : energy per beam section area, $E_{\text{surf}} = dE/dS$.

Power, “puissance” $P$ [W] (power of a light beam, pulse peak power) : energy crossing the beam section by time unit, $P = dE/dt$.

Intensity, “intensité” $I$ [W.m$^{-2}$] : power density per surface area unit, power per beam section area, average value of the modulus of the Poynting vector, $I = dP/dS$.

Photon flux, “Flux de photons” $\Phi$ [m$^{-2}.s^{-1}$] : number of photons $^1$ of frequency $\nu$ crossing a unit area surface per second, $\Phi = I/h\nu$.

$^1$In this course, the word ‘photon’ is used only for its convenience. Indeed, since here we treat the electromagnetic field classically, it represents only an amount of energy $h\nu$ inside the considered cavity.
1.2. THE LASER PROCESS

**Energy density**, “densité d’énergie” $u \text{[J.m}^{-3}\text{]}$: energy of the electromagnetic wave per unit volume.

For a plane or spherical progressive wave, the following relation can be easily established:

$$I = \frac{c_0}{n_0} u,$$

(1.1)

where $c_0/n_0$ is the wave velocity in the considered medium of refractive index $n_0$.

Let us consider for example a monochromatic plane wave of angular frequency $\omega = 2\pi \nu$ propagating along the $+z$ direction. The electric field associated with this wave reads:

$$E(r,t) = -\vec{e} A e^{-i\omega t+ikz} + \text{c.c.},$$

(1.2)

where $A$ is complex and where $\vec{e}$ is a unit vector, supposed to be real for the sake of simplicity, and oriented along the polarization of the field. The energy density of this wave is given by:

$$u = \frac{1}{2} (E \cdot D + B \cdot H),$$

(1.3)

where $E$ is the electric field, $D$ the electric displacement, $B$ the magnetic induction and $H$ the magnetic field. Equation (1.3) becomes, in our case:

$$u = 2\varepsilon_0 n_0^2 |A|^2.$$

(1.4)

The intensity $I$ is given by the modulus of the Poynting vector

$$S = E \wedge H,$$

(1.5)

leading to

$$I = 2\varepsilon_0 n_0 c_0 |A|^2.$$

(1.6)

One can check that equations (1.4) and (1.6) are compatible with (1.1).

1.2.2 Energy conversion processes

Bohr’s hypothesis (1913) states that the total energy of an “atom” (which can also be a molecule, an ion, etc...) can exhibit only discrete values, and that it can be modified only through “quantum jumps” accompanied by, for example, the emission or the absorption of a photon. This particular
energy conversion process is described by Einstein’s theory of the interaction of matter and electromagnetic radiation (1917).

In the case of a system of two levels labeled 1 and 2 of respective energies $E_1$ and $E_2$ ($E_1 < E_2$), we can describe five main energy conversion schemes (see figure 1.2):

- (a) Indirect excitation of levels 1 and 2, also called pumping. This pumping can be performed by bringing energy to the system under various forms (light, chemical energy, electrical energy, collisions,...). This is usually achieved via other levels of the atom.

- (b) The atom decays from level 2 to level 1 by spontaneous emission of a photon of energy $h\nu_0 = E_2 - E_1$: this emission at the resonance frequency $\nu_0$ of the transition $2 \rightarrow 1$ occurs in any spatial direction ($4\pi$ sr), usually isotropically. It can be anisotropic in some crystalline solids. Once level 2 is excited, the moment of this quantum jump is random, but the spontaneous emission probability follows an exponentially decreasing law versus time. This finite lifetime of level 2 leads to a broadening of the emission profile centered at the resonance frequency $\nu_0$. This process cannot be rigorously described without quantifying the electromagnetic field. Consequently, within the framework of this course, we will introduce spontaneous emission in a heuristic manner.

- (c) The atom, initially in level 1, is excited to level 2 by absorption of a photon of energy $h\nu$, where the frequency $\nu$ of the incoming radiation is very close to the resonance frequency $\nu_0$ of the transition $2 \rightarrow 1$. 

Figure 1.2: Schematic representation of energy conversion processes between atoms and radiation.
1.2. THE LASER PROCESS

This resonant process is induced by the incident photon. Each time this process occurs, the incoming light loses a photon: it is attenuated (or absorbed) after having experienced many such processes.

- (d) The atom, initially in level 2, decays to level 1 by stimulated (or induced) emission of a photon of energy $h\nu$, where the frequency $\nu$ of the incoming light is very close to the resonance frequency $\nu_0$ of the transition $2 \to 1$. This process, which is induced (or stimulated) by the incident photon, is exactly the reverse of the absorption process: the incident radiation gains a photon which has exactly the same characteristics (frequency, direction, ...) as the incident ones. The incident light is amplified after having experienced many such processes.

- (e) The atom, initially in level 2, decays to level 1 by non radiative decay. No photon is emitted, and the energy is transferred in a non radiative manner: collisions, vibrational or rotational excitation of a molecule, emission of phonons in a solid, ...

All these processes occur in a laser amplifying medium. Stimulated emission (d), which is the basis of the amplification, can occur only if a sufficient number of atoms are excited to level 2 by the pumping process (a), so that process (d) is more likely to occur than absorption (c). We will see that this can occur only in a situation called population inversion, i.e., when the population of level 2 is larger than the one of level 1. Spontaneous emission (b) allows the “first photon” to be emitted, which will then be amplified by stimulated emission. Non radiative decays (e) are usually irreversible and rather efficient, and can be useful to efficiently empty level 1 or to pump level 2 by decay from other levels.

1.2.3 Heuristic derivation of the equations of the single-frequency laser

The concepts developed above permit us to imagine a simplified description of the laser effect. Let us suppose that it relies on the $2 \to 1$ transition. In the most favorable case, the lower level 1 is not the fundamental level of the atom and can quickly decay by (b) and even more efficiently by (e) to lower-lying levels: this is the so-called “four-level system”. If one is able to use process (a) to efficiently pump the upper level (2), it is sensible to admit that the number of atoms in level 1 is negligible compared to the number of atoms in level 2. Within this approximation (which is valid for many types of lasers), one can describe the laser operation by two quantities: the total
1. LIGHT-MATTER INTERACTION

(dimensionless) number $\Delta N$ of atoms in level 2 in the amplifying medium and the number $F$ of photons circulating inside the laser cavity.

The number $F$ evolves mainly under the influence of process (d), and also because of the unavoidable intracavity losses (mirror losses, diffusion, residual absorption, diffraction,...). The differential equation for $F$ reads:

$$\frac{dF}{dt} = \kappa \Delta NF - \frac{F}{\tau_{\text{cav}}}. \quad (1.7)$$

The term $-F/\tau_{\text{cav}}$ corresponds to the decay of the number of intracavity photons induced by the cavity losses and is characterized by the time constant $\tau_{\text{cav}}$ called the “photon lifetime”. The product term $\kappa \Delta NF$ is the rate of increase of the number of photons (a.k.a. the gain) due to stimulated emission. This rate is proportional to the number of atoms in level 2 and to the number of photons in the laser cavity.

If one wants to introduce spontaneous emission, one can as a first approximation replace $\kappa \Delta NF$ by $\kappa \Delta N(F + 1)$. However, usually, $F \approx 10^{10} - 10^{20}$, and the contribution of spontaneous emission to this term can be neglected because $F \gg 1$.

The number $\Delta N$ changes mainly under the influence of processes (a) which excites atoms to level 2, (b) which makes the atoms decay out of level 2 to level 1 and other levels, and (d) which makes atoms decay to level 1 at a rate proportional to $F$. The differential equation governing the evolution of $\Delta N(t)$ can be written:

$$\frac{d\Delta N}{dt} = R_p - \kappa \Delta NF - \frac{\Delta N}{\tau}. \quad (1.8)$$

The term $R_p$ holds for the pumping rate to level 2. The term $-\Delta N/\tau$ is the spontaneous decay rate of the atoms in level 2. $\tau$ is called the “lifetime of level 2”. $\kappa \Delta NF$ is the decay rate of level 2 due to stimulated emission, which is proportional to the number of intracavity photons. It is worth noticing that the constant $\kappa$ is the same in equations (1.7) and (1.8), because the decay of one atom out of level 2 by stimulated emission (1.8) corresponds to the emission of one photon in the cavity (1.7). This constant $\kappa$ will be explicitly determined later.

The two important equations (1.7) et (1.8), also called the Statz and de Mars equations, describe the laser behavior, provided the quantities $R_p$, $\kappa$, $\tau$, and $\tau_{\text{cav}}$ are known. They constitute a set of nonlinear coupled equations which do not generally have an analytical solution. However they can be easily solved in some particular cases (see chapters 3 and 4).
Comment: these equations are not valid for a three-level system in which the lower level labeled 1 has a long lifetime or is the fundamental level of the considered atom. Moreover, they assume that the laser operates in single-frequency regime. We will develop those assumptions later.

1.2.4 Steady-state regime: saturation and threshold

The steady-state regime, which corresponds to the continuous wave (cw) operation of the laser, corresponds to the following simple equations: $d\Delta N/dt = 0$ and $dF/dt = 0$. Equations (1.7) and (1.8) together with the hypothesis $F \neq 0$ then lead to the two following equations:

\[
\Delta N = \frac{1}{\kappa \tau_{\text{cav}}} \equiv \Delta N_{\text{th}},
\]

\[
F = \frac{1}{\kappa \Delta N_{\text{th}}} \left( R_p - \frac{\Delta N_{\text{th}}}{\tau} \right).
\]

The quantity $\Delta N_{\text{th}}$ defined in (1.9) is a constant depending on the characteristics of the considered transition and of the cavity losses: it is the number of atoms present in level 2 when the laser oscillates in cw regime. It does not depend on the laser intensity. In particular, it also corresponds to the number of atoms at the laser threshold, when its intensity is almost zero (the subscript ‘th’ in $\Delta N_{\text{th}}$ holds for threshold). When $\Delta N < \Delta N_{\text{th}}$, there is no laser operation in cw regime. Figure 1.3 summarizes the steady-state operation of the laser with respect to the pumping rate.

Let us introduce the pumping rate at threshold:

\[
R_{p}^{\text{th}} = \frac{1}{\kappa \tau_{\text{cav}}},
\]

the equation (1.10) shows that $F$ evolves linearly with the pumping rate $R_p$ once the laser is above threshold:

\[
F = \frac{1}{\kappa \Delta N_{\text{th}}} \left( R_p - \frac{\Delta N_{\text{th}}}{\tau} \right) = \tau_{\text{cav}} \left[ R_p - R_{p}^{\text{th}} \right], \quad \text{if } R_p \geq R_{p}^{\text{th}},
\]

\[
F = 0, \quad \text{if } R_p < R_{p}^{\text{th}}.
\]
The equation $R_{p}^{th} = \Delta N_{th}/\tau$ simply means that the pumping rate at threshold exactly compensates the decay rate per atom times the number of atoms in level 2 at threshold. Equation (1.10) also evidences the influence of the laser radiation on the number of atoms in the excited level 2. It can indeed equivalently be written as:

$$\Delta N = \frac{R_{p}\tau}{1 + \kappa\tau F} = \frac{R_{p}\tau}{1 + \frac{F}{F_{sat}}}$$

with $F_{sat} = \frac{1}{\kappa\tau}$.

(1.14)

The number $F_{sat}$ defined in (1.14) is a constant depending on the characteristics of the transition. In the absence of laser oscillation ($F = 0$), the number of atoms in level 2 would be equal to $\Delta N_{0} = R_{p}\tau$. The “saturation photon number” is defined to be the number of photons $F = F_{sat}$ for which the number of atoms in level 2 is divided by 2, i.e., for which $\Delta N = \Delta N_{0}/2$. $\Delta N$ is called the “saturated” population of level 2, $\Delta N_{0}$ the “unsaturated” population of level 2.

One can notice in (1.14) that saturation is a nonlinear phenomenon. It plays a major role because it explains how the number of atoms in level 2 is, above threshold, independent of the value of $F$. Notice also that when $F$ becomes very large, $\Delta N$ tends to 0. Actually, the increase of the pump (and consequently of $\Delta N_{0}$) leads to an increase of the number of intracavity photons while $\Delta N$ remains constant.

We will come back later in chapter 3 to the laser operation in steady-state regime. In the next section of the present chapter, we are going to justify...
the expressions of the gain and saturation mechanisms in a more rigorous manner.

1.3 Optical Bloch equations for a two-level system

1.3.1 Electric-dipole interaction . Rabi frequency

Let us consider a two-level atom. The two levels $|1\rangle$ and $|2\rangle$, which we suppose, for now, to be non degenerate, have respective energies $E_1$ and $E_2 = E_1 + \hbar \omega_0$ (see figure 1.4). If the atom’s Hamiltonian is $\hat{H}$, the evolution of the state $|\Psi\rangle$ of the atom is given by Schrödinger’s equation:

$$i\hbar \frac{d|\Psi\rangle}{dt} = \hat{H}|\Psi\rangle.$$  \hspace{1cm} (1.15)

Let us investigate the interaction of the atom with a monochromatic electromagnetic field of angular frequency $\omega$ and linearly polarized along $Ox$. The corresponding field reads:

$$E = E_x u_x = (A e^{-i\omega t} + \text{c.c.}) u_x,$$  \hspace{1cm} (1.16)

where $u_x$ is a unit vector along $x$, $E_x$ the real instantaneous amplitude of the field and $A$ its complex amplitude at the atom location. The Hamiltonian $\hat{H}$ is given by $\hat{H} = \hat{H}_0 + \hat{H}_I$ where $\hat{H}_0$ is the Hamiltonian of the free atom and where $\hat{H}_I$ is the electric-dipole interaction Hamiltonian:

$$\hat{H}_I = -\hat{d} \cdot E = -\hat{d}_x E_x,$$  \hspace{1cm} (1.17)

where $\hat{d}$ is the electric dipole operator and $\hat{d}_x$ its component along the direction $x$. Let us introduce the matrix element for this operator between
levels 1 and 2: \(d_{12} = \langle 1 | \hat{d} | 2 \rangle\) (the diagonal elements of this operator are zero for an atom in vacuum). We suppose that \(d_{12}'\) is real, allowing us to write \(d \equiv d_{12} = d_{21}\).

State \(|\Psi\rangle\) can be expanded in the \(|1\rangle, |2\rangle\) basis:
\[
|\Psi\rangle = a_1(t) \exp \left(-i \frac{E_1 t}{\hbar}\right) |1\rangle + a_2(t) \exp \left(-i \frac{E_2 t}{\hbar}\right) |2\rangle.
\]
By injecting the expansion (1.18) into (1.15) with (1.17), one gets:
\[
\dot{a}_1 = i \frac{1}{2} a_2 \left( \omega_1 \exp(-i\omega t) + c.c. \exp(i\omega t) \right),
\]
\[
\dot{a}_2 = i \frac{1}{2} a_1 \left( \omega_1 \exp(-i\omega t) + c.c. \exp(i\omega t) \right),
\]
where we have introduced the complex Rabi angular frequency:
\[
\omega_1 = \frac{2dA}{\hbar}.
\]
In equations (1.19) and (1.20), we keep only the slowly varying terms (this is the so-called quasi-resonant approximation, also called “rotating wave approximation” or “secular approximation”), leading to:
\[
\dot{a}_1 = i \frac{1}{2} a_2 \omega_1^* \exp(i\delta t),
\]
\[
\dot{a}_2 = i \frac{1}{2} a_1 \omega_1 \exp(-i\delta t),
\]
where we have introduced the detuning between the wave frequency and the atomic transition Bohr frequency:
\[
\delta = \omega - \omega_0.
\]
Combining equations (1.22) and (1.23) leads to the equation of evolution of \(a_1\):
\[
\ddot{a}_1 - i\delta \dot{a}_1 + \frac{1}{4} |\omega_1|^2 a_1 = 0.
\]
Its solutions oscillate at the nutation angular frequency \(\Omega\) given by
\[
\Omega = \sqrt{|\omega_1|^2 + \delta^2}.
\]
One obtains the well known Rabi oscillations which are studied in matter-wave interaction courses.
1.3. OPTICAL BLOCH EQUATIONS

1.3.2 Density Matrix (or Density Operator)

The formalism using the Schrödinger equation is valid only for an isolated atom. If one wants to introduce relaxation mechanisms and allow some statistical dispersion among the considered atoms, the density matrix formalism must be used. The Schrödinger equation is then replaced by Von Neumann’s equation:

\[ i\hbar \frac{d}{dt} \rho = \{H, \rho\} . \]  

(1.27)

In the case of a two-level atom, we obtain:

\[ \dot{\rho}_{11} = -\dot{\rho}_{22} = \frac{i}{2} (\omega_1 e^{-i\omega t} + \omega_1^* e^{i\omega t}) (\rho_{21} - \rho_{12}) , \]  

(1.28)

\[ \dot{\rho}_{12} = i\omega_0 \rho_{12} + \frac{i}{2} (\omega_1 e^{-i\omega t} + \omega_1^* e^{i\omega t}) (\rho_{22} - \rho_{11}) . \]  

(1.29)

Using again the quasi-resonant approximation, this set of equations becomes:

\[ \dot{\rho}_{11} = -\dot{\rho}_{22} = \frac{i}{2} (\omega_1^* \rho_{21} e^{i\omega t} - \omega_1 \rho_{12} e^{-i\omega t}) , \]  

(1.30)

\[ \dot{\rho}_{12} = i\omega_0 \rho_{12} + \frac{i\omega_1}{2} (\rho_{22} - \rho_{11}) e^{i\omega t} . \]  

(1.31)

We then move to the frame rotating in the complex plane with the field of complex amplitude \[ A = |A| e^{i\phi} \] by introducing:

\[ \sigma_{21}(t) = \rho_{21}(t) e^{i\omega t} , \]  

(1.32)

\[ \sigma_{12}(t) = \rho_{12}(t) e^{-i\omega t} . \]  

(1.33)

This allows us to obtain the Bloch equations for our two-level system:

\[ \frac{d}{dt} (\rho_{22} - \rho_{11}) = -i (\omega_1^* \sigma_{21} - \omega_1 \sigma_{21}^* ) , \]  

(1.34)

\[ \frac{d}{dt} \sigma_{21} = i\delta \sigma_{21} - \frac{i\omega_1}{2} (\rho_{22} - \rho_{11}) . \]  

(1.35)

The Bloch vector \((u, v, w)\) is usually introduced using these equations in treatises dealing with light-matter interaction. As far as we are concerned, we are going to follow a slightly different path: since a laser is a dissipative system, relaxations are key elements. We thus choose now to introduce them phenomenologically into the optical Bloch equations.
1.3.3 Relaxation. Pumping

The levels $|1\rangle$ and $|2\rangle$ of our laser transition exhibit, in general, finite lifetimes $\tau_1$ and $\tau_2$, respectively (the case where level 1 is the fundamental level will be dealt with in chapter 2). We thus introduce (see figure 1.5), the relaxation rates $\gamma_1 = 1/\tau_1$ and $\gamma_2 = 1/\tau_2$ for the populations $\rho_{11}$ and $\rho_{22}$. Part of the population decaying from level 2 feeds level 1, for example by spontaneous emission. Let us call $A$ the associated decay rate. The coherence lifetime is usually much shorter than the population lifetime. We call $\Gamma$ the decay rate of the coherences $\rho_{12}$. One has $\Gamma \geq (\gamma_1 + \gamma_2)/2$. The pumping to levels 1 and 2 is taken into account by the pumping rates $\Lambda_1$ and $\Lambda_2$, respectively. Introducing these different relaxation and pumping rates into equations (1.30) and (1.31), the Bloch equations of our two-level system become:

\[
\begin{align*}
\frac{d}{dt}\rho_{22} &= \Lambda_2 - \gamma_2 \rho_{22} - \frac{i}{2} (\omega_1^* \sigma_{21} - \omega_1 \sigma_{21}^*) , \\
\frac{d}{dt}\rho_{11} &= \Lambda_1 - \gamma_1 \rho_{11} + A \rho_{22} + \frac{i}{2} (\omega_1^* \sigma_{21} - \omega_1 \sigma_{21}^*) , \\
\frac{d}{dt}\sigma_{21} &= -(\Gamma - i\delta) \sigma_{21} - \frac{i}{2} \omega_1 (\rho_{22} - \rho_{11}) .
\end{align*}
\]
1.4 Steady-state regime. Gain. Saturation

1.4.1 Steady-state regime

In steady-state regime, the solutions of equations (1.36-1.38) will allow us to determine the susceptibility of the medium. We start by deriving the state of the system in the absence of laser field ($\omega_1 = 0$).

1.4.1.1 Without the laser field

In this case ($\omega_1 = 0$), the system has the following steady-state solution:

\[ \sigma_{21}^{(0)} = \sigma_{12}^{(0)} = 0, \]
\[ \rho_{22}^{(0)} - \rho_{11}^{(0)} = \frac{\Lambda_2 (\gamma_1 - A) - \Lambda_1 \gamma_2}{\gamma_1 \gamma_2}. \] (1.39)

In particular, one can notice that in order to get a population inversion ($\rho_{22} > \rho_{11}$), we must have $\gamma_1 > A$. In general, in laser materials, we have $\gamma_1 \gg A$. In this case, the population inversion is obtained when $\Lambda_2 \gamma_1 > \Lambda_1 \gamma_2$, meaning that the upper level is pumped more efficiently than the lower level.

1.4.1.2 With the laser field

When $\omega_1 \neq 0$, the steady-state solutions of equations (1.36-1.38) become:

\[ \sigma_{21} = \frac{\omega_1}{2} (\rho_{22} - \rho_{11}) \frac{\delta - i\Gamma}{\Gamma^2 + \delta^2}, \]
\[ \rho_{22} - \rho_{11} = \frac{\rho_{22}^{(0)} - \rho_{11}^{(0)}}{1 + \frac{\gamma_1 + \gamma_2 - A |\omega_1|^2}{\gamma_1 \gamma_2} \frac{\Gamma}{\Gamma^2 + \delta^2}}. \] (1.41)

1.4.2 Susceptibility. Saturation

In the case of a plane wave, we introduce

\[ I_{\text{sat}} = \frac{\varepsilon_0 c n_0 \hbar^2}{d^2 \Gamma} \frac{\gamma_1 \gamma_2}{\gamma_1 + \gamma_2 - A} (\Gamma^2 + \delta^2), \] (1.43)

and, using equations (1.6) and (1.21), equation (1.42) reads:

\[ \rho_{22} - \rho_{11} = \frac{\rho_{22}^{(0)} - \rho_{11}^{(0)}}{1 + \frac{I}{I_{\text{sat}}}}. \] (1.44)
We can thus see that when the intensity of the wave becomes much stronger than the saturation intensity $I_{\text{sat}}$, the system becomes transparent and does no longer respond to the incident wave.

The response of the system to the laser field is given, in steady-state regime, by its susceptibility which can be obtained thanks to the polarization created by the wave in the atomic medium. This macroscopic polarization is given by:

$$P_{\text{at},x} = n\langle p_x \rangle = n\text{Tr}\{\rho \hat{d}_x\} = 2nd\text{Re}\rho_{21}, \quad (1.45)$$

where $n$ is the atomic density. This polarization $P_{\text{at},x}$ is just the part of the polarization of the active medium which comes from the active atoms. The polarization of the matrix in which the active atoms are embedded is given by:

$$P_{\text{mat},x} = \varepsilon_0 \chi_{\text{mat}} A e^{-i\omega t} + \text{c.c.}, \quad (1.46)$$

where $n_0 = \sqrt{1 + \chi_{\text{mat}}}$ is the refractive index of the matrix at frequency $\omega$. The total polarization of the active medium is given by $P_x = P_{\text{mat},x} + P_{\text{at},x}$. Using equation (1.32), we obtain:

$$P_{\text{at},x} = 2nd \text{Re}\{\sigma_{21} e^{-i\omega t}\} \quad (1.47)$$

Besides, the susceptibility $\chi_{\text{at}} = \chi'_{\text{at}} + i\chi''_{\text{at}}$ for the atoms at frequency $\omega$ is defined by:

$$P_{\text{at},x} = \varepsilon_0 \chi_{\text{at}} A e^{-i\omega t} + \text{c.c.}. \quad (1.48)$$

By comparing equations (1.47) and (1.48), one gets:

$$\chi'_{\text{at}} = \frac{nd^2}{\varepsilon_0 \hbar} (\rho_{22}^{(0)} - \rho_{11}^{(0)}) \frac{1}{1 + \frac{I}{I_{\text{sat}}(\delta)}} \frac{\delta}{\Gamma^2 + \delta^2}, \quad (1.49)$$

$$\chi''_{\text{at}} = -\frac{nd^2}{\varepsilon_0 \hbar} (\rho_{22}^{(0)} - \rho_{11}^{(0)}) \frac{1}{1 + \frac{I}{I_{\text{sat}}(\delta)}} \frac{\Gamma}{\Gamma^2 + \delta^2}. \quad (1.50)$$

Consequently, $\chi''$ exhibits a Lorentzian profile and $\chi'$ the associated dispersion profile with $\chi' (\delta) = -\frac{\delta}{\Gamma} \chi'' (\delta)$. Let us notice that

$$\frac{1}{1 + \frac{I}{I_{\text{sat}}(\delta)}} \frac{1}{\Gamma^2 + \delta^2} = \frac{1}{\Gamma^2 \left(1 + \frac{I}{I_{\text{sat}}(\delta=0)}\right) + \delta^2}. \quad (1.51)$$

We can see that the saturation broadens the Lorentzian by a factor $\sqrt{1 + I/I_{\text{sat}}(\delta = 0)}$. This is the so-called saturation broadening. Similarly, when $I/I_{\text{sat}}$ increases, the real and imaginary parts of the susceptibility decrease and tend to 0: at strong intensities, the atoms do no longer interact with the field.
Let us also mention the fact that
\[ I_{\text{sat}}(\delta) = I_{\text{sat}}(\delta = 0) \frac{\Gamma^2 + \delta^2}{\Gamma^2}, \] (1.52)
illustrating the fact that when the detuning increases, the interaction of the atoms with the field gets less efficient.

1.4.3 Gain. Dispersion

Let us remind the influence of this susceptibility on the wave propagation. We consider a plane wave propagating along \( z \) with an amplitude \( E = A e^{i\omega t + ikz} + \text{c.c.} \). This wave must obey the Helmholtz equation:
\[ \frac{\partial^2 E}{\partial z^2} = \frac{1}{c_0^2} (1 + \chi) \frac{\partial^2 E}{\partial t^2}, \] (1.53)
where \( \chi = \chi_{\text{mat}} + \chi_{\text{at}} \) is the total susceptibility of the active medium. We thus have
\[ k = \frac{\omega}{c_0} (1 + \chi)^{1/2} \simeq \frac{n_0 \omega}{c_0} \left( 1 + \frac{\chi'_{\text{at}}}{2n_0^2} + i \frac{\chi''_{\text{at}}}{2n_0^2} \right). \] (1.54)
The plane wave thus reads:
\[ E = A e^{-i\omega t} \exp \left[ i \frac{n_0 \omega}{c_0} \left( 1 + \frac{\chi'_{\text{at}}}{2n_0^2} \right) z \right] \exp \left[ -\frac{n_0 \omega}{c_0} \frac{\chi''_{\text{at}}}{2n_0^2} z \right] + \text{c.c.}. \] (1.55)
We can thus conclude that the wave propagates with a refractive index \( n_0 \left( 1 + \frac{\chi'_{\text{at}}}{2n_0^2} \right) \) (dispersion) and experiences a \textit{gain per unit length} coefficient \( \alpha \) defined by \( I(z) = I(0) e^{\alpha z} \) and thus given by:
\[ \alpha = -\frac{\omega}{n_0 c_0} \chi''_{\text{at}}. \] (1.56)
Let us remind that, in steady-state regime, for a plane wave propagating along the +z direction, we have:
\[ \alpha = \frac{1}{I} \frac{dI}{dz}. \] (1.57)
We can thus see that light is amplified when \( \alpha > 0 \), i. e., when \( \chi''_{\text{at}} < 0 \), i. e., when, according to (1.50), \( \rho_{22} > \rho_{11} \):
\[ \text{Gain} > 0 \iff \text{Population inversion}. \] (1.58)

\(^2\)For a wave propagating along the –z direction, the gain per unit length would be given by \( \alpha = -\frac{1}{I} \frac{dI}{dz} \).
1. LIGHT-MATTER INTERACTION

1.5 Laser cross section. Generalizations

1.5.1 Definition of the cross section

The laser cross section is defined with respect to the gain coefficient per unit length according to the following equation:

\[ \alpha(\nu) \equiv \sigma(\nu)\Delta n, \]  

(1.59)

where the population inversion (per unit volume) is given by:

\[ \Delta n = n(\rho_2 - \rho_1) = n_2 - n_1. \]  

(1.60)

\(n_2\) and \(n_1\) are the populations (per unit volume) of levels 2 and 1, respectively.

The saturation intensity \(I_{\text{sat}}\) of equation (1.43) can be written as:

\[ I_{\text{sat}}(\nu) = \frac{h\nu_0}{\sigma(\nu)\tau}, \]  

(1.61)

provided one defines the response time of the system\(^4\) \(\tau\) according to:

\[ \tau = \tau_2 + \tau_1 - A\tau_1\tau_2. \]  

(1.62)

We hence see that the gain can be written:

\[ \alpha(\nu) = \frac{\alpha_0(\nu)}{1 + I/I_{\text{sat}}(\nu)} = \sigma(\nu)\Delta n = \frac{\sigma(\nu)\Delta n_0}{1 + I/I_{\text{sat}}(\nu)}, \]  

(1.63)

where \(\alpha_0\) is the unsaturated gain coefficient and \(\Delta n_0\) the unsaturated population inversion. \(\alpha\) and \(\Delta n\) are the saturated gain saturated population inversion, respectively. The usefulness of the concept of cross section introduced in (1.59) lies in the fact that it allows to make the difference between the characteristics of the transition (which lie in the value of \(\sigma\) and its frequency dependence) and the status of the populations described by \(\Delta n\).

Up to now, we have considered only the case of an elementary two-level system. Of course, most laser media cannot be modeled in such a simple manner. We must consequently generalize the preceding results along several directions.

\(^3\)In this course, we shift without notice from a dependance of several quantities, such as, e. g., \(\sigma\), in detuning \(\delta\) to a dependance in frequency \(\nu\) or in angular frequency \(\omega\). In order to avoid to get lost, one just needs to remember that \(\delta = \omega - \omega_0 = 2\pi(\nu - \nu_0)\).

\(^4\)The definitions of \(I_{\text{sat}}\) (1.61) and \(\tau\) (1.62) given in this chapter are valid only for atomic systems in which the lower level of the transition is not the fundamental level (see figure 1.5). The opposite case, called the three-level system, will be dealt with in chapters 2 and 3.
1.5. LASER CROSS SECTION

1.5.2 Generalization to degenerate levels

Contrary to the system we have just considered, it happens that levels 1 and 2 are degenerate. We call $g_1$ and $g_2$ their respective degeneracies. In this case, the global stimulated emission and absorption result from the sum over all the possible transitions between the different sublevels. The preceding results, and more precisely equations (1.59) and (1.61), can be easily generalized provided the population inversion be defined by:

$$
\Delta n = n_2 - \frac{g_2}{g_1} n_1,
$$

(1.64)

where $n_1$ and $n_2$ are the total populations (per unit volume) in levels 1 and 2, respectively.

1.5.3 Generalization to more complicated profiles

Up to now, we have considered only transitions for which the width is due only to the finite lifetimes $\tau_1$ and $\tau_2$ of the levels and $1/\Gamma$ of the optical coherences. In this case, we have seen (see equation 1.50) that the transition exhibits a Lorentzian profile which can be described by:

$$
\sigma(\nu) = \sigma(\nu_0) \frac{\pi \Delta \nu}{2} \frac{1}{L(\nu - \nu_0)} .
$$

(1.65)

where the line center lies at $\nu_0$ and where we have introduced the normalized Lorentzian profile of Full Width at Half Maximum (FWHM) $\Delta \nu$:

$$
L(\nu - \nu_0) = \frac{2}{\pi \Delta \nu} \frac{1}{1 + [2(\nu - \nu_0)/\Delta \nu]^2} .
$$

(1.66)

This profile is normalized to have a unit area:

$$
\int_{-\infty}^{+\infty} L(\nu - \nu_0) d\nu = 1.
$$

(1.67)

All lines do not exhibit Lorentzian profiles. For example, when the transition comes from the addition of several transitions between non degenerate sublevels, its profile can exhibit a very different shape. This occurs for example in the case of ions experiencing the Stark effect created by the crystalline field due to the matrix in which they are embedded. In this case, the normalized profile is called $g(\nu - \nu_0)$ and one has:

$$
\sigma(\nu) = \sigma(\nu_0) \frac{g(\nu - \nu_0)}{g(0)} = \sigma_0 \frac{g(\nu - \nu_0)}{g(0)} ,
$$

(1.68)
where $\sigma_0$ is the cross section at line center and where we have

$$\int_{-\infty}^{+\infty} g(\nu - \nu_0) d\nu = 1.$$  \hfill (1.69)

Let us finally mention the fact that up to now we have considered only situations where all the atoms are identical and thus have the same laser cross section $\sigma(\nu)$. Such a medium is said to be homogeneously broadened. The opposite case in which the atoms have different spectral characteristics (inhomogeneously broadened medium) also exists and will be the subject of chapter 4.
Chapter 2

Equations of the single-frequency laser

In this chapter, starting from the optical Bloch equations, we derive the dynamical equations describing the single-frequency laser based on a homogeneously broadened medium. We then introduce the concept of adiabatic elimination and discuss the different laser classes. We then obtain the famous “rate equations” of the laser.

2.1 Bloch-Maxwell equations of the laser

We start here from Bloch’s equations (1.36-1.38), which describe the
atoms of the active medium, in order to derive the equations of evolution of a single-frequency laser. We consider a ring cavity as sketched in figure 2.1. We also suppose that the active medium is spatially homogeneous and fills the whole cavity. We call $z$ the abscissa along the light propagation axis inside the cavity and we suppose that the intracavity field can be treated as a plane wave, namely:

$$E(z,t) = A(z,t)e^{-i(\omega t - kz)} + A^*(z,t)e^{i(\omega t - kz)} = 2 \text{Re} \left[ A(z,t)e^{-i(\omega t - kz)} \right] ,$$  

where we have supposed that the polarization of the intracavity field is fixed, allowing us to treat light as a scalar quantity. $A$ is called the slowly variable complex amplitude, meaning that it depends on $t$ and $z$, but in a much slower way than the plane wave term $e^{-i(\omega t - kz)}$. We suppose in this chapter that the laser oscillates in a single direction and at a single frequency $\omega$.

### 2.1.1 Equation of evolution for the polarization

In a manner similar to equation (2.1), the polarization of the active atoms can be written:

$$P_{at}(z,t) = P(z,t)e^{-i(\omega t - kz)} + P^*(z,t)e^{i(\omega t - kz)} = 2 \text{Re} \left[ P(z,t)e^{-i(\omega t - kz)} \right] .$$  

Equation (1.45) then becomes:

$$P(z,t) = n d \sigma_{21}(z,t) .$$  

Notice that we have abandoned the subscript ‘at’ for the slowly variable complex amplitude $P$. We deduce the equation of evolution of $P$ from equations (1.38) and (1.21):

$$\frac{d}{dt} P = - (\Gamma - i\delta) P - \frac{i d^2}{\hbar} A \Delta n .$$  

### 2.1.2 Equation of evolution of the population inversion

In order to simplify the calculations, we restrict our derivation to the case of the so-called “four-level system” (see section 2.3) in which the lower level decays very fast ($\gamma_1 \gg \gamma_2$) and is not pumped ($\Lambda_1 \ll \Lambda_2$). In this case, we can consider that the lower level is always empty ($\rho_{11} = 0$), and the population inversion per unit volume is simply given by:

$$\Delta n = n_2 = n \rho_{22} .$$
2.1. BLOCH-MAXWELL EQUATIONS

Equation (1.36) becomes:

\[
\frac{d}{dt} \Delta n = -\frac{1}{\tau} (\Delta n - \Delta n_0) - \frac{i}{\hbar} (A^* P - AP^*) ,
\]

where, in this case, the response time of the system \( \tau \) is simply given by the lifetime of the upper level of the transition \( \tau = \tau_2 = 1/\gamma_2 \) and where the pumping term has been recast into the form \( \Delta n_0/\tau \) with

\[
\Delta n_0 = n \Lambda_2 \tau .
\]

2.1.3 Equation of evolution of the field

The intracavity field \( E \) is a solution of Maxwell’s equations and must consequently obey:

\[
\frac{\partial^2 E}{\partial z^2} - \varepsilon_0 \mu_0 \frac{\partial^2 E}{\partial t^2} - \varsigma \mu_0 \frac{\partial E}{\partial t} = \mu_0 \frac{\partial^2 P}{\partial t^2} ,
\]

where the cavity losses are introduced via a fictitious conductivity \( \varsigma \) uniformly distributed in the cavity (see below). The polarization \( P = P_{\text{mat}} + P_{\text{at}} \) contains the source term \( P_{\text{at}} \) which creates the electromagnetic field. Consequently, we can see that the atomic coherences are responsible for the emission of the laser field. Replacing \( E \) and \( P_{\text{at}} \) by their expressions (2.1) and (2.2) leads to the equation of evolution of the field slowly variable amplitude:

\[
\frac{c_0}{n_0} \frac{\partial A}{\partial z} + \frac{\partial A}{\partial t} + \varsigma \frac{A}{2\varepsilon} = i \frac{\omega}{2\varepsilon} P ,
\]

where we have used the slowly variable amplitude approximation which is valid provided that the gain and losses are weak enough to justify the following approximations:

\[
\begin{align*}
\left| \frac{\partial^2 A}{\partial z^2} \right| & \ll k \left| \frac{\partial A}{\partial z} \right| , \\
\left| \frac{\partial^2 A}{\partial t^2} \right| & \ll \omega \left| \frac{\partial A}{\partial t} \right| , \\
\left| \frac{\partial^2 P}{\partial t^2} \right| & \ll \omega \left| \frac{\partial P}{\partial t} \right| \ll \omega \varepsilon \left| \frac{\partial A}{\partial t} \right| , \\
\left| \frac{\partial A}{\partial t} \right| & \ll \omega |A| .
\end{align*}
\]

At this point, we suppose that the cavity losses are weak, allowing us to neglect the dependence of \( A \) on \( z \) and thus to neglect the term \( \frac{c_0}{n_0} \frac{\partial A}{\partial z} \) with
2. EQUATIONS OF THE SINGLE-FREQUENCY LASER

Figure 2.2: Losses in the cavity of length $L_{\text{cav}}$. The reflection and transmission coefficients of mirrors $M_i$ ($i = 1, 2, 3$) are noted $R_i$ and $T_i$, respectively. The extra cavity losses (absorption, diffusion,...) are noted $\eta$.

respect to the term $\frac{\partial A}{\partial t}$ in equation (2.9). The latter can thus be rewritten as:

$$\frac{dA}{dt} = -\frac{1}{2\tau_{\text{cav}}} A + \frac{i\omega}{2\epsilon} P,$$

(2.14)

where we have introduced the lifetime of the photons in the cavity $\tau_{\text{cav}} = \epsilon/\varsigma$. Finally, let us take into account the fact that the angular frequency $\omega$ at which we have expanded the field in equation (2.1) is detuned from the cavity resonance frequency $\omega_q$ by a quantity $\delta_{\text{cav}} = \omega - \omega_q$, leading to:

$$\frac{dA}{dt} = -\left(\frac{1}{2\tau_{\text{cav}}} - i\delta_{\text{cav}}\right) A + \frac{i\omega}{2\epsilon} P,$$

(2.15)

2.1.4 Cavity losses and detuning

The first term on the right-hand side of equation (2.15) can be understood by considering the cold (i.e., without gain) cavity, as shown in figure 2.2. At time $t$, let us consider the intracavity field $A(z = 0, t)$ at origin $A$ of abscissa $z$. After one round-trip in the cavity, i.e., after a time $L_{\text{cav}}n_0/c_0$, the field at point $A$ is

$$A\left(z = 0, t + \frac{L_{\text{cav}}n_0}{c_0}\right) = \sqrt{R_1R_2R_3(1-\eta)}A(z = 0, t)e^{ikL_{\text{cav}}},$$

(2.16)

where the coefficients $R_i$ are the intensity reflection coefficients of the mirrors and where $\eta$ holds for the other cavity losses (diffusion, residual absorption
from the active medium and the intracavity optical components, diffraction losses,...). One can thus see that the phase accumulated per round-trip is 

\[ kL_{\text{cav}} = \frac{n_0}{c_0} L_{\text{cav}} \omega \equiv \frac{n_0}{c_0} L_{\text{cav}} \delta_{\text{cav}} \, [2\pi] \]

because the resonance pulsations of the cavity \( \omega_q = q \frac{n_0 L_{\text{cav}}}{c_0} \), where \( q \) is an integer, precisely check \( \frac{n_0}{c_0} L_{\text{cav}} \omega_q \equiv 0 \, [2\pi] \). This explains the term \(-i \delta_{\text{cav}} A\) in equation (2.15).

By using the fact that

\[ \left| A \left( z = 0, t + \frac{L_{\text{cav}} n_0}{c_0} \right) \right| = \sqrt{R_1 R_2 R_3 (1 - \eta)} |A(z = 0, t)| , \]  

(2.17)

and, since the round-trip losses are supposed to be weak, that

\[ \left| A \left( z = 0, t + \frac{L_{\text{cav}} n_0}{c_0} \right) \right| \simeq |A(z = 0, t)| + \frac{L_{\text{cav}} n_0}{c_0} \frac{d|A|}{dt} , \]  

(2.18)

we are led to

\[ \frac{d|A|}{dt} = - \frac{|A|}{2\tau_{\text{cav}}} , \]  

(2.19)

with

\[ \tau_{\text{cav}} = \frac{n_0 L_{\text{cav}}}{c_0} \left[ 1 - \sqrt{R_1 R_2 R_3 (1 - \eta)} \right]^{-1} . \]  

(2.20)

We eventually obtain the following expression for the lifetime of the photons in the cavity:

\[ \tau_{\text{cav}} \simeq \frac{n_0 L_{\text{cav}}}{c_0 (1 - R_1) + (1 - R_2) + (1 - R_3) + \eta} = \frac{\text{Duration of a cavity round-trip}}{\text{Losses per cavity round-trip}} . \]  

(2.21)

We also define the quality factor \( Q_{\text{cav}} \) of the cavity:

\[ Q_{\text{cav}} = \omega_0 \frac{\text{stored energy}}{\text{dissipated power}} = \omega_0 \frac{I}{dI/dt} = \omega_0 \tau_{\text{cav}} . \]  

(2.22)

Of course, in the case of a linear cavity such as the one of figure 1.1, the length \( L_{\text{cav}} \) corresponds to twice the cavity length.

### 2.2 Adiabatic elimination. Laser classes

#### 2.2.1 Discussion. Laser dynamic variables

The Maxwell-Bloch equations governing the behavior of our single-frequency laser based on a four-level system are summarized below:
2. EQUATIONS OF THE SINGLE-FREQUENCY LASER

\[
\begin{align*}
\frac{dA}{dt} &= -\left(\frac{1}{2\tau_{\text{cav}}} - i\delta_{\text{cav}}\right)A + i\frac{\omega}{2\epsilon}P, \quad (2.23) \\
\frac{dP}{dt} &= -(\Gamma - i\delta)P - i\frac{\hbar}{\epsilon}A\Delta n, \quad (2.24) \\
\frac{d\Delta n}{dt} &= -\frac{1}{\tau}(\Delta n - \Delta n_0) - i\frac{\hbar}{\epsilon}(A^*P - AP^*). \quad (2.25)
\end{align*}
\]

These equations perfectly illustrate the mechanism of the interaction between the electromagnetic field and the two-level system. Equation (2.24) shows that the field induces a polarization (i.e., atomic coherences) in the two-level system when this latter presents a population difference between the two levels ($\Delta n \neq 0$). The phase of this induced polarization depends on the phase of the field, with a sign that changes depending whether $\Delta n$ is positive or negative. Then, as shown by equation (2.23), this polarization will emit a field which will either lead to a global amplification (stimulated emission) or attenuation (absorption) of the field, depending on its sign. Moreover, this field will act on the atoms which have a non-zero polarization to alter their populations (see equation 2.25).

Our laser is consequently a dynamical system presenting three dynamical variables: the field, the population inversion, and the atomic polarization. These three variables are nonlinearly coupled, as can be seen from the last terms in equations (2.24) and (2.25). Such a set of differential equations, since it is nonlinear and presents at least three dynamical variables, can exhibit complicated dynamical behaviors such as quasi-periodicity or deterministic chaos. Such things may happen in the case of lasers: it can indeed be shown that equations (2.23-2.25) are formally equivalent to Lorentz’s equations that led to the discovery of the famous “butterfly effect”. However, such a chaotic behavior happens only in very unusual lasers, as will be discussed below.

2.2.2 Lasers dynamic classes

The three ordinary differential equations (2.23-2.25) have some similarities. In particular, they all contain a relaxation term with a lifetime given by $\tau_{\text{cav}}$, $\tau$ or $\Gamma^{-1}$ for the field, population inversion, or the polarization, respectively. These times are characteristic of the time needed by each variable to reach a new steady-state value after a parameter has been changed. Depending on the respective values of these three time scales, the laser may belong to one of the three following classes:

**Class C lasers**: $\tau_{\text{cav}}$, $\tau$, and $\Gamma^{-1}$ are all of the same order of magnitude.
The main example of such lasers is the NH\textsubscript{3} laser which operates in the far infrared. It has been shown to exhibit deterministic chaos.

**Class B lasers**: \( \tau_{\text{cav}}, \tau \gg \Gamma^{-1} \) and \( \tau_{\text{cav}} \) is not much larger than \( \tau \). The ruby, Nd:YAG, diode lasers and some CO\textsubscript{2} lasers belong to this class. In this case, the atomic polarization \( P \) responds very quickly to the changes of \( \Delta n \) and of the field \( A \). We may thus suppose that \( P \) reaches its steady-state value instantaneously and write \( dP/dt = 0 \) in equation (2.24). This allows to express \( P \) as a function of \( \Delta n \) and \( A \) and to eliminate it from the set of equations (2.23-2.25). This is the so-called process of **adiabatic elimination** which will be also discussed in the following paragraph.

**Class A lasers**: \( \tau_{\text{cav}} \gg \tau, \Gamma^{-1} \). Most gas lasers and dye lasers belong to this class. In this case, we can adiabatically eliminate \( P \) and \( \Delta n \), whose response times are much shorter than the response time of the intracavity field.

It is worth noticing that these three classes of lasers have the same steady-state solutions. However, their transient behaviors are different and depend on the class to which they belong.

In the following, we shall focus our attention on class B and class A lasers, which cover most lasers, and discuss their steady-state and transient behaviors.

### 2.2.3 Adiabatic elimination of the polarization

Let us thus suppose that we are in the situation where \( \tau_{\text{cav}}, \tau \gg \Gamma^{-1} \). We can thus write \( dP/dt = 0 \) in equation (2.24), allowing us to replace at every instant \( P(t) \) by its steady-state value:

\[
P(t) = -\frac{i\hbar^2}{\hbar} \frac{1}{\Gamma} A(t) \Delta n(t).
\]  

By injecting this result into equation (2.25) and using (1.6) and (1.43), one obtains the equation of evolution of the population inversion:

\[
\frac{d}{dt} \Delta n = \frac{1}{\tau} \left( \Delta n_0 - \Delta n - \frac{I}{I_{\text{sat}}} \Delta n \right).
\]  

Similarly, using (1.6), one gets:

\[
\frac{dI}{dt} = 2\varepsilon_0 n_0 c_0 \left( A^* \frac{dA}{dt} + \text{c.c.} \right).
\]
Thanks to equations (2.23) and (2.26), we can get the equation of evolution for the intensity:

\[
\frac{dI}{dt} = \frac{I}{\tau_{\text{cav}}} \left( \frac{\Delta n}{\Delta n_{\text{th}}} - 1 \right),
\]

where

\[
\Delta n_{\text{th}} = \frac{n_0}{\sigma\tau_{\text{cav}} c_0} = \frac{\Pi}{\sigma L_{\text{cav}}},
\]

and where \( \Pi \) holds for the intensity losses per cavity round-trip.

Equations (2.27) and (2.29) constitute the so-called Statz and de Mars equations. The adiabatic elimination of the polarisation, permitting to describe the state of the atoms through their populations only, is often called the “rate equation” approximation. We will make this approximation in all the rest of this course.

### 2.2.4 Equations of evolution in terms of numbers of photons and atoms

It is worth rewriting equations (2.27) and (2.29) by replacing the intensity of the wave and the density of population inversion by the number of photons inside the cavity and the (dimensionless) number of atoms in the excited level in the laser mode volume. To this aim, we suppose that the intracavity laser beam has a uniform section area \( S \) throughout the cavity. Then, the number of photons inside the cavity reads:

\[
F = \frac{I}{h\nu} \frac{n_0 L_{\text{cav}}}{c_0} S,
\]

and the population inversion expressed in terms of number of atoms (which, in the present case, corresponds to the number of atoms in the upper level) becomes:

\[
\Delta N = V_{\text{cav}} \Delta n,
\]

where we have defined the volume occupied by the laser mode in the active medium

\[
V_{\text{cav}} = L_{\text{cav}} S.
\]

With these new notations, equations (2.27) and (2.29) become:

\[
\frac{dF}{dt} = -\frac{F}{\tau_{\text{cav}}} + \kappa F \Delta N,
\]

\[
\frac{d}{dt} \Delta N = -\frac{1}{\tau} (\Delta N - \Delta N_0) - \kappa F \Delta N,
\]
2.2. LASER CLASSES

with

\[ \kappa = \frac{c_0}{n_0} \frac{\sigma}{V_{cav}} \]  
(2.36)

2.2.5 Case where the active medium does not fill the cavity

Figure 2.3: Laser cavity containing an active medium which does not fill it up.

In the case where the active medium does not fill the cavity (see figure 2.3), equations (2.34) and (2.36) remain valid with the following changes:

\[ \tau_{cav} = \frac{L_{cav,opt}/c_0}{\Pi} = \frac{\text{Duration of a cavity round-trip}}{\text{Losses per round-trip}} \]  
(2.37)

where \( L_{cav,opt} \) is the optical length of the cavity, \( \Pi \) the losses per cavity round-trip, and:

\[ \Delta N = V_a \Delta n \]  
(2.38)

where \( V_a \) is the volume occupied by the laser mode in the active medium.

The same term \( \kappa F \Delta N \) is present in both equations (2.34) and (2.35), showing that when a photon is emitted by stimulated emission, an atom gets de-excited.
Up to now, we have restricted our discussion to the simple case of a four-level system. This has allowed us to rigorously justify the laser equations which had been heuristically obtained in chapter 1 (see equations 1.7 and 1.8). The general case will be dealt with in what follows, where the rate equations in the general case are going to be written.

### 2.3 Rate equations. Three- and four-level systems

#### 2.3.1 Rate equation approximation: general case

The approximation used till the end of the present chapter is the so-called "rate equation approximation." These equations describe only the exchange of energy quanta between matter and light without taking into account the phase coherence of the electromagnetic field nor the atomic coherences. They are based on the transition probabilities and are sufficient to describe the oscillation of a single-mode laser.

For the levels 1 and 2 (which are supposed to be non-degenerate) of the amplifying transition in interaction with the laser field, the equations describing the time evolution of the dimensionless populations $N_2$ and $N_1$ of levels 1 and 2 read:

$$\frac{dN_2}{dt} = -\sum_{i \neq 2} \left( \gamma_{2i}^r + \gamma_{2i}^n \right) N_2 - \kappa FN_2 + \sum_{j \neq 2} \left( \gamma_{j2}^r + \gamma_{j2}^n \right) N_j + \kappa FN_1,$$

$$\frac{dN_1}{dt} = -\sum_{i \neq 1} \left( \gamma_{1i}^r + \gamma_{1i}^n \right) N_1 - \kappa FN_1 + \sum_{j \neq 1} \left( \gamma_{j1}^r + \gamma_{j1}^n \right) N_j + \kappa FN_2,$$

where the coefficients $\gamma_{ij}^r$ and $\gamma_{ij}^n$ are the radiative and non-radiative decay rates of level $i$ to level $j$. One can similarly write the decay rates of the levels which play a role in the pumping process and introduce the pumping rates. Solving such sets of equations is usually impossible. It is thus useful to establish simpler models that could reproduce the experimental results. The two extreme cases that occur are the so-called three- and four-level systems that we are now going to present in details. Of course, many experimental situations correspond to intermediate cases involving a larger number of levels.

#### 2.3.2 Three-level system

A three-level system is a system in which level 1 is the fundamental level of the considered atom. Level 2 is often (but not always!) the first excited level,
and level 3 is an intermediate level used to pump level 2. We suppose that the decay of level 3 is fast enough to make its population negligible \(N_3 \approx 0\). Figure 2.4 summarizes the notations used to describe this three-level system, which is supposed to be closed \((N_1 + N_2 = N)\). In the case where levels 1 and 2 are non degenerate we have:

\[
\frac{dN_2}{dt} = W_P N_1 - AN_2 - \kappa F \Delta N , \tag{2.41}
\]
\[
\frac{dN_1}{dt} = -W_P N_1 + AN_2 + \kappa F \Delta N , \tag{2.42}
\]

where \(W_P\) is the pumping probability per unit time and where \(A\) is the spontaneous emission probability per unit time. Using these two equations with the fact that \(N_1 + N_2 = N\) leads to:

\[
\frac{d\Delta N}{dt} = (W_P - A)N - (W_P + A)\Delta N - 2\kappa F \Delta N . \tag{2.43}
\]

Equation (2.43) can be re-written in the more usual form:

\[
\frac{d}{dt} \Delta N = \frac{1}{\tau} \left( \Delta N_0 - \Delta N - \frac{I}{I_{sat}} \Delta N \right) , \tag{2.44}
\]

provided the response time of the system, the pumping rate, and the saturation intensity are defined according to:

\[
\tau = (W_P + A)^{-1} , \tag{2.45}
\]
\[
\Delta N_0 = N \frac{W_P - A}{W_P + A} , \tag{2.46}
\]
\[
I_{sat} = \frac{h \nu}{2 \sigma \tau} . \tag{2.47}
\]
In the absence of field \((I = 0)\), the steady-state solution of equation (2.44) is \(\Delta N = \Delta N_0\). Equation (2.46) shows that for \(W_P = 0\) one has \(\Delta N_0 = -N\) (all the atoms are in the lower level) and the medium absorbs light. In order to “bleach” the system (i.e., to make it transparent), one must apply a minimum pumping rate \(W_P = A\). The system can be turned into an amplifier (\(\Delta N_0 > 0\)) only when the pumping rate exceeds \(A\), as shown in figure 2.5. Moreover, one notices a factor 2 at the denominator of equation (2.47). This corresponds to a decrease of the saturation intensity by a factor of 2 with respect to the four-level system. It is due to the fact that each emitted photon decreases \(N_2\) by 1, increases \(N_1\) by 1, and thus decreases \(\Delta N\) by 2. In terms of photon numbers and number of inverted atoms, the equations of a laser based on a three-level system thus become:

\[
\frac{dF}{dt} = -\frac{F}{\tau_{\text{cav}}} + \kappa F \Delta N ,
\]

\[
\frac{d}{dt} \Delta N = -\frac{1}{\tau}(\Delta N - \Delta N_0) - 2\kappa F \Delta N ,
\]

2.3.3 Four-level system

A four-level system is a system in which level 0 is the fundamental level of the atom, and in which levels 1 and 2 are both excited levels. Level 3 is an excited
level used to pump level 2. We suppose that levels 1 and 3 decay fast enough to make their populations negligible \((N_1 \approx N_3 \approx 0)\). Figure 2.6 summarizes the notations. The system is supposed to be closed \((N_0 + N_2 = N)\). If we suppose again that levels 1 and 2 are not degenerate, we get:

\[
\frac{dN_2}{dt} = W_P N_0 - AN_2 - \kappa FN_2 , \\
\frac{dN_0}{dt} = -W_P N_0 + AN_2 + \kappa FN_2 .
\]

Using these two equations and the fact that \(N_0 + N_2 = N\) we have:

\[
\frac{d\Delta N}{dt} = W_P N - (W_P + A)\Delta N - \kappa F\Delta N .
\]

Equation (2.52) can be written according to the usual formulation

\[
\frac{d}{dt} \Delta N = \frac{1}{\tau} \left( \Delta N_0 - \Delta N - \frac{I}{I_{sat}} \Delta N \right) .
\]

leading to the following expressions of the response time of the system, the pumping rate, and the saturation intensity:

\[
\tau = (W_P + A)^{-1} , \\
\Delta N_0 = N \frac{W_P}{W_P + A} , \\
I_{sat} = \frac{h\nu}{\sigma \tau} .
\]
In the absence of field \( I = 0 \), the steady-state solution of equation (2.53) is \( \Delta N = \Delta N_0 \). Equation (2.55) shows that for \( W_P = 0 \) one has \( \Delta N_0 = 0 \) (all the atoms are in the lower level) and the medium is transparent for the laser. As soon as \( W_P > 0 \), the medium exhibits gain \( (\Delta N > 0) \), as shown in figure 2.5. Moreover, there is no factor of 2 at the denominator of equation (2.56). Indeed, the stimulated emission of a photon decreases \( \Delta N \) only by 1 because level 1 remains always empty. In terms of number of photons and number of inverted atoms, the equations of a laser based on a four-level system become:

\[
\frac{dF}{dt} = -\frac{F}{\tau_{\text{cav}}} + \kappa F \Delta N , \tag{2.57}
\]

\[
\frac{d}{dt} \Delta N = -\frac{1}{\tau} (\Delta N - \Delta N_0) - \kappa F \Delta N , \tag{2.58}
\]

### 2.3.4 Standard equations

In order to write laser equations which are valid in both cases, we introduce the factor \( 2^* \) which is equal to 1 for a four-level system and 2 for a three-level system, leading to:

\[
\frac{dF}{dt} = -\frac{F}{\tau_{\text{cav}}} + \kappa F \Delta N , \tag{2.59}
\]

\[
\frac{d}{dt} \Delta N = -\frac{1}{\tau} (\Delta N - \Delta N_0) - 2^* \kappa F \Delta N , \tag{2.60}
\]

### 2.3.5 Introduction of spontaneous emission

One can show, using the tools of quantum optics, the spontaneous emission falling in the laser mode corresponds to an emission rate which would be the one induced by one photon inside the cavity. We can thus take into account to replace equation (2.59) by the following equation:

\[
\frac{dF}{dt} = -\frac{F}{\tau_{\text{cav}}} + \kappa [(F + 1)N_2 - FN_1] . \tag{2.61}
\]
2.3. RATE EQUATIONS

2.3.6 Pumping mechanisms

**Radiative pumping** (a.k.a. “optical pumping”) using flash lamps, arc lamps, the sun, ion lasers (Ar\(^+\), Kr\(^+\)), nitrogen lasers, excimer lasers (XeCl, KrF...), diode lasers, etc. Examples: ruby laser (Cr\(^{3+}\):Al\(_2\)O\(_3\)), titanium-sapphire laser (Ti\(^{3+}\)Al\(_2\)O\(_3\)), alexandrite laser (Cr\(^{3+}\):BeAl\(_2\)O\(_4\)), neodymium-YAG laser (Nd\(^{3+}\):Y\(_3\)Al\(_5\)O\(_12\)), neodymium-glass laser, dye laser (rhodamines, coumarins, stilbene, fluorescein, pyridine, oxazine...), color-center laser, etc.

**Electronic pumping** using D.C. or radiofrequency electric discharges in gases (helium-neon laser, helium-cadmium laser, Ar\(^+\) laser, Kr\(^+\) laser, nitrogen laser, excimer lasers, copper vapor laser, CO\(_2\) laser,...) or electron beams (free-electron laser).

**Thermal pumping** by hydrodynamic expansion (CO\(_2\) lasers, CO lasers,...).

**Chemical pumping** by exothermic chemical combustion or by fast combustion (I, HF, HCl, HBr, CO lasers,...).

**Pumping by injection of carriers**: bias current in the p-n junction of a semiconductor diode laser (violet to infrared diode lasers: PbSnTe, PbSSe, GaInAsSb, GaInAsP, GaAlAs, GaInP, ZnCdSSe, GaN).

**Pumping by heavy particles or by ionizing radiations**: ion beams, fission products, X-ray sources, nuclear explosions (!),...
2. EQUATIONS OF THE SINGLE-FREQUENCY LASER
Chapter 3

Single-frequency laser in steady-state regime

In this chapter, we use the equations of the single-frequency laser derived in the preceding chapter to study the steady-state solutions. We also discuss some extensions of these equations. Let us recall that we suppose that the gain medium is homogeneously broadened.

3.1 Steady-state solutions

3.1.1 Determination of the steady-state solutions for $F$ and $\Delta N$

Let us look for the steady-state solutions of the set of equations derived in chapter 2:

$$\frac{dF}{dt} = -\frac{F}{\tau_{cav}} + \kappa F \Delta N ,$$

(3.1)

$$\frac{d\Delta N}{dt} = -\frac{1}{\tau}(\Delta N - \Delta N_0) - 2^* \kappa F \Delta N ,$$

(3.2)

where, as shown above, $2^*$ is equal to 1 for a four-level system and 2 for a three-level system. Taking $\frac{dF}{dt} = 0$ and $\frac{d\Delta N}{dt} = 0$ in these equations, we find the two steady-state solutions:

3.1.1.1 ‘OFF’ solution

The first solution corresponds to
3. STEADY-STATE REGIME

\[ \Delta N_{\text{OFF}} = \Delta N_0, \quad (3.3) \]
\[ F_{\text{OFF}} = 0. \quad (3.4) \]

It corresponds to the laser turned off. There is no light in the cavity and the population inversion is not saturated.

### 3.1.1.2 ‘ON’ solution

The second solution corresponds to

\[
\Delta N_{\text{ON}} = \frac{1}{\kappa \tau_{\text{cav}}} = \frac{S \Pi}{\sigma} = \Delta N_{\text{th}}, \quad (3.5)
\]
\[
F_{\text{ON}} = \frac{1}{2^* \kappa T} \left( \frac{\Delta N_0}{\Delta N_{\text{th}}} - 1 \right) = \frac{1}{2^* \kappa T} (r - 1), \quad (3.6)
\]

where \( \Pi \) are the losses per round-trip and where the relative excitation \( r \) has been defined according to

\[ r = \frac{\Delta N_0}{\Delta N_{\text{th}}}. \quad (3.7) \]

Using equation (1.6), equation (3.6) becomes:

\[ I_{\text{ON}} = I_{\text{sat}} (r - 1). \quad (3.8) \]

This solution corresponds to the laser turned on. It is worth noticing that the value of the population inversion remains blocked to its value at threshold \( \Delta N_{\text{th}} \) for any value of the pumping, i.e., of the unsaturated population inversion \( \Delta N_0 \). This is easily understood by rewriting equations (3.7) and (3.8) in the following manner:

\[ \Delta N_{\text{th}} = \frac{\Delta N_0}{1 + I/I_{\text{sat}}}. \quad (3.9) \]

One can thus see that once the laser is on, the population inversion gets saturated by the intracavity intensity and remains blocked to the value \( \Delta N_{\text{th}} \).

### 3.1.2 Stability of the steady-state solutions

We can see that there exists two steady-state solutions, the so-called ‘ON’ and ‘OFF’ solutions, for any value of \( \Delta N_0 \). In the following, we analyze their stability to know which solution will be chosen by the laser.
3.1. STEADY-STATE SOLUTIONS

3.1.2.1 Stability of the ‘OFF’ solution

Let us suppose that we move the laser slightly away from the “OFF” solution:

\[ \Delta N(t) = \Delta N_{OFF} + x(t) = \Delta N_0 + x(t) \]  
\[ F(t) = F_{OFF} + y(t) = y(t) \]

where \( x(t) \) et \( y(t) \) are small quantities. By injecting (3.10) and (3.11) into the equations (3.1) and (3.2) for the evolution of the laser, and keeping only first-order terms one gets:

\[ \dot{x}(t) = -\frac{x(t)}{\tau} - 2^*\kappa\Delta N_0 y(t) \]  
\[ \dot{y}(t) = (\kappa\Delta N_0 - \frac{1}{\tau_{cav}}) y(t) \]

which can be rewritten:

\[ \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = M \begin{pmatrix} x \\ y \end{pmatrix} \]

The matrix \( M \) is given by

\[ M = \begin{pmatrix} -1/\tau & -2^*\kappa\Delta N_0 \\ 0 & \kappa\Delta N_0 - 1/\tau_{cav} \end{pmatrix} \]

The eigenvalues of \( M \) are \(-1/\tau\) and \(\kappa\Delta N_0 - 1/\tau_{cav}\). The solution ‘OFF’ will consequently be stable if the real parts of these two eigenvalues are both negative, i. e., as long as \(\Delta N_0 < 1/\kappa\tau_{cav} = \Delta N_{th}\). The laser is then said to be below threshold. The gain term \(\kappa F\Delta N\) is smaller than the losses \(-F/\tau_{cav}\) in the equation of evolution (3.1) of the intensity. Above threshold, the solution ‘OFF’ is no longer stable. In summary:

\[
\text{Laser threshold } \iff \Delta N_0 = \Delta N_{th} = \frac{1}{\kappa\tau_{cav}} \iff \text{Unsaturated gain } = \text{losses}.
\]

Indeed, using (3.5), the gain at threshold is given by

\[ \alpha_{th} L_a = \sigma \Delta n_{th} L_a = \sigma \frac{\Delta N_{th}}{L_a S} L_a = \Pi, \]

where \( L_a \) is the length of the active medium.
Figure 3.1: Steady-state solutions of the laser versus unsaturated population inversion. Full line: stable solution; dotted line: unstable solution.

3.1.2.2 Stability of the ‘ON’ solution

Once the laser oscillates, equation (3.9) becomes valid. It can be summarized in the following manner:

\[
\text{Laser on } \iff \Delta N = \Delta N_{\text{th}} = \frac{1}{\kappa T_{\text{cav}}} \iff \text{Saturated gain } = \text{losses}
\]  

Indeed, the equality of the gain and losses terms in equation (3.1) precisely shows that this corresponds to a steady-state solution because the saturated gain is then equal to the losses. To make sure that this solution is stable above threshold, let us move the laser slightly away from the ‘ON’ solution:

\[
\Delta N(t) = \Delta N_{\text{ON}} + x(t) = \Delta N_{\text{th}} + x(t),
\]

\[
F(t) = F_{\text{ON}} + y(t),
\]

where \(x(t)\) and \(y(t)\) are supposed to be small. By injecting (3.19) and (3.20) in equations (3.1) and (3.2) and keeping only first-order terms, one gets an equation similar to (3.14) where the matrix \(M\) is given by

\[
M = \begin{pmatrix}
-\frac{r}{\tau} & -\frac{2^*}{\tau_{\text{cav}}} \\
\frac{1}{(r-1)/2^*\tau} & 0
\end{pmatrix}
\]  

(3.21)
The eigenvalues of this matrix are:

\[ \lambda_{\pm} = -\frac{r}{2\tau} \pm \frac{1}{2} \sqrt{\left(\frac{r}{\tau}\right)^2 - \frac{4(r-1)}{\tau \tau_{\text{cav}}}}. \]  

(3.22)

One can see that as soon as \( r < 1 \) the solution \( \lambda_+ \) becomes positive, showing that the ‘ON’ solution is unstable. On the contrary, when \( r > 1 \), the real parts of the two eigenvalues are always negative, evidencing the stability of the ‘ON’ solution.

3.1.3 Summary. Comparison between three- and four-level systems

Figure 3.1 summarizes the preceding discussion about the steady-state solutions of the laser. In particular, it shows that the threshold actually corresponds to the transition from one steady-state solution to the other occurring when the unsaturated gain exceeds the losses. This evolution is plotted versus the unsaturated population inversion \( \Delta N_0 \). As seen earlier, \( \Delta N_0 \) does not evolve versus the pumping rate \( W_p \) in the same manner in a three-level and in a four-level system (see subsection 2.3). If we imagine that we can find a three-level system and a four-level system exhibiting identical parameters, figure 3.2 compares their behaviors versus \( W_p \). One can see that the three-level system presents two disadvantages:

1. An important amount of pumping is used to bleach the medium before gain can be obtained and
2. Since \( I_{\text{sat}} \) is twice smaller for a three-level system than for a four-level system, the slope of the evolution of the power versus pumping is also twice smaller.

3.2 Laser frequency

In steady-state regime, we have determined up to now only the intensity of the laser. But, of course, light is also characterized by its frequency and its phase, which we are going to discuss now.

3.2.1 Cavity modes. Frequency Pulling

3.2.1.1 Field self-consistency

Let us consider a laser cavity such as the one of figure 2.1 or figure 2.2, in which the active medium completely fills the cavity. If we suppose again

\(^1\)Anyway, we should not forget that the first laser, namely the ruby laser, is a nice example of three-level system.
3. STEADY-STATE REGIME

Figure 3.2: Compared evolutions versus pumping rate $W_P$ of (a) the unsaturated population inversion $\Delta N_0$, (b) the population inversion $\Delta N$ and (c) the number of photons $F$ for a three-level and a four-level system.
that the laser oscillates in only one direction, in a single-frequency regime, and that the intracavity beam can be described by a plane wave, then the intracavity electromagnetic field reads:

\[ E(z, t) = A e^{-i \omega t + ikz} + \text{c.c.} \quad (3.23) \]

The fact that the laser is in steady-state regime is equivalent to the fact that the field is equal to itself after propagation through one cavity round-trip, i.e.:

\[ \sqrt{R_1 R_2 R_3 (1 - \eta)} e^{ikL_{\text{cav}}} = 1, \quad (3.24) \]

where \( R_1, R_2, \) and \( R_3 \) are the reflection coefficients of the cavity mirrors and \( \eta \) holds for the other losses.

In equation (3.24), \( k \) is complex and can be written, as seen in chapter 1:

\[ k = n_0 \omega \left( 1 + \frac{\chi_{\text{at}}}{2n_0^2} \right) + i \frac{\omega}{c_0} \frac{\chi''_{\text{at}}}{2n_0^2} = n_0 \omega \left( 1 + \frac{\chi_{\text{at}}}{2n_0^2} \right) - i \frac{\alpha}{2}. \quad (3.25) \]

### 3.2.1.2 Field module

Taking the module of equation (3.24), we get:

\[ \ln [R_1 R_2 R_3 (1 - \eta)] = -\alpha L_{\text{cav}} , \quad (3.26) \]

which is equivalent to:

\[ \alpha L_{\text{cav}} = \Pi. \quad (3.27) \]

We just find that, in steady-state regime, the saturated gain of the active medium exactly compensates the losses.

### 3.2.1.3 Field frequency

Taking the argument of equation (3.24), one gets:

\[ \frac{n_0 \omega}{c_0} \left( 1 + \frac{\chi_{\text{at}}}{2n_0^2} \right) L_{\text{cav}} = 2q\pi , \quad (3.28) \]

where \( q \) is an integer. The frequency \( \nu \) of the laser is thus given by:

\[ \nu \left( 1 + \frac{\chi_{\text{at}}(\nu)}{2n_0^2} \right) = q \frac{c_0}{n_0 L_{\text{cav}}} = \nu_q , \quad (3.29) \]

where \( \nu_q \) is the so-called “empty” (i.e., without the gain) cavity resonance frequency. We can thus see that the active medium modifies the laser frequency with respect to the resonance frequency \( \nu_q \) of the empty cavity. Let
us suppose for example that our active medium has a Lorentzian gain with a FWHM $\Delta \nu$. Then, using equations (1.49), (1.50) and (1.56), we have:

$$\chi_\text{at}'(\nu) = -\frac{\nu - \nu_0}{\Delta \nu/2} \chi_\text{at}''(\nu) = \frac{\nu - \nu_0}{\Delta \nu/2} \frac{n_0 c_0}{\omega_0} \alpha(\nu).$$

(3.30)

If we suppose that $\nu$ is close to $\nu_q$, the frequency shift created by the amplifying atoms reads:

$$\nu - \nu_q \simeq -\frac{\chi_\text{at}'(\nu_q)}{2n_0^2} \nu_q,$$

(3.31)

leading to

$$\frac{\nu - \nu_q}{\nu_q} \simeq \frac{\nu_0 - \nu_q}{\Delta \nu} \frac{c_0}{n_0 \omega_0} \alpha(\nu_q).$$

(3.32)

We can thus see that the frequency shift is respectively positive or negative depending whether $\nu_q$ is smaller or larger than the center frequency $\nu_0$ of the transition. The active medium thus “pulls” the frequency $\nu$ towards the gain maximum: this is the so-called “frequency pulling” effect.

In the case where the active medium has a length $L_a$ and does not completely fill the cavity, equations (3.27) and (3.32) become:

$$\alpha L_a = \Pi,$$

(3.33)

$$\frac{\nu - \nu_q}{\nu_q} \simeq \frac{\nu_0 - \nu_q}{\Delta \nu} \frac{c_0}{n_0 \omega} \frac{L_a \alpha(\nu_q)}{L_{\text{cav}}}.$$

(3.34)

Let us introduce the quality factors $Q_{\text{cav}}$ and $Q_a$ for the cavity and the active medium, respectively:

$$Q_{\text{cav}} = \omega \tau_{\text{cav}} = \frac{\omega}{c_0} \frac{L_{\text{cav, opt}}}{\Pi},$$

(3.35)

$$Q_a = \frac{\nu_0}{\Delta \nu}.$$

(3.36)

Equation (3.34) then reads:

$$\nu = \frac{Q_{\text{cav}} \nu_q + Q_a \nu_0}{Q_{\text{cav}} + Q_a}.$$

(3.37)

This equation could also have been obtained by taking the argument of equation (2.23) in steady-state regime.
3.2. LASER FREQUENCY

3.2.1.4 Phase of the field

Equation (3.24) has allowed us to determine the intensity and the frequency of the intracavity field (3.23). However, it does not impose the phase of the field, i.e., the argument of $A$. Consequently, one can see that the phase of the field can take any value. When the laser is turned on, the first “photons” choose an arbitrary phase and the field is later amplified by stimulated emission. If one switches the laser off and starts the same experiment again, the laser will reach the same intensity and frequency but a different phase. However, we will see in chapter 6 that spontaneous emission then induces a random walk for the phase.

3.2.2 Single-frequency operation

Figure 3.3: Frequency range in which the laser may oscillate. The unsaturated gain $\alpha_0$ is plotted versus frequency $\nu$. $\Pi L_\alpha$ is the corresponding losses level. The modes whose unsaturated gain is larger than the losses are the only ones that can oscillate.

Up to now, we have just supposed that our laser based on a homogeneously broadened medium was monomode. This means that only one of the cavity modes at frequency $\nu_q$ oscillates. However, one can prove that if the active medium broadening is strictly homogeneous and if the laser exhibits no “spatial hole burning” (see section 3.4), this is rigorously true. Let us first determine how many longitudinal modes may in principle oscillate in
our laser. To this aim, we must count the number of modes that may be above threshold, i.e., those for which the unsaturated gain is larger than the losses. Figure 3.3 shows an example in which two modes are above threshold. Indeed, the modes labeled \( q \) and \( q + 1 \) experience an unsaturated gain larger than the losses and lie in the possible oscillation range for that laser. The other modes are below threshold.

Figure 3.4: Laser unsaturated gain \( \alpha_0 \) versus frequency. (a) The gain is below losses for all the frequencies: the laser cannot oscillate. (b) The gain is larger than the losses at frequency \( v_0 \) but smaller than the losses at the cavity mode frequencies: the laser cannot oscillate. (c) Mode \( q + 1 \) reaches threshold and starts to oscillate. (d) The two modes \( q \) and \( q + 1 \) are above threshold \( (\alpha_0(v_q) > \Pi/L_a \) and \( \alpha_0(v_{q+1}) > \Pi/L_a) \). However, once mode \( q + 1 \) oscillates, the saturated gain \( \alpha(v_q) \) is lower than the losses and does not allow the mode \( q \) to oscillate.

Let us thus suppose now that these two modes of frequencies \( v_q \) and \( v_{q+1} \) are the only modes able to oscillate and let us examine what happens when we increase the unsaturated gain \( \alpha_0(\nu) \), for example by increasing the pumping. As long as the unsaturated gain \( \alpha_0 \) is smaller than the losses for all the cavity frequencies \( v_q \), the laser is below threshold and does not oscillate [see figures 3.4(a) and 3.4(b)]. The laser reaches threshold when the unsaturated gain of one of the modes is equal to the losses [see figure 3.4(c)]. For stronger pumping rates [see figure 3.4(d)], the gain is saturated by the oscillating mode and becomes smaller than the losses for all the other modes, who can consequently no longer oscillate. The laser is thus always monomode and oscillates on the mode which has the strongest unsaturated gain.
3.3 Laser power

3.3.1 Optimal output coupling

The steady-state solutions of the laser equations allow us to determine the number of photons and consequently the intensity inside the laser cavity. However, one is most often interested in using the output laser beam and thus in optimizing the output power by choosing the optimum output coupler transmission coefficient. Let us consider for example the laser of figure 3.5,

![Laser diagram](image)

Figure 3.5: Laser with an output coupling with transmission $T$.

whose output coupling mirror has a transmission $T$. We suppose that all the other losses (other mirrors, diffusion, absorption,...) correspond to an amount of intensity losses per round-trip $\Upsilon = \Pi - T$. Then, using equation (3.8), the intracavity intensity is

$$I = I_{\text{sat}}(r - 1) = I_{\text{sat}} \left( \frac{\alpha_0 L_a}{T + \Upsilon} - 1 \right), \quad (3.38)$$

where $L_a$ is the length of the active medium. The laser output intensity is thus given by:

$$I_{\text{out}} = TI_{\text{sat}} \left( \frac{\alpha_0 L_a}{T + \Upsilon} - 1 \right). \quad (3.39)$$

The evolution of $I_{\text{out}}$ versus $T$ is plotted in figure 3.6. The laser oscillates for $0 \leq T \leq T_{\text{max}} = \alpha_0 L_{\text{cav}} - \Upsilon$. The maximum output power $I_{\text{out}}^{\text{max}}$ is obtained for the optimal transmission $T_{\text{opt}}$. They are given by:
In the limit where the pumping is very strong and the laser is far above threshold ($\alpha_0 L_a \gg \Upsilon$), the maximum laser output power is given by:

$$P_{\text{out}}^{\text{max}} \simeq S I_{\text{sat}} \alpha_0 L_a = \frac{h \nu \Delta N_0}{\tau}.$$  (3.42)

We can thus see that the maximum power that one can extract from the laser medium is equal to the energy that can be stored in the active medium ($h \nu \Delta N_0$) divided by the gain recovery time $\tau$.

### 3.3.2 Power in the vicinity of threshold

As mentioned above, the laser starts from spontaneous emission. If we want to predict how the laser power depends on the excitation ratio $r$ close to threshold $r = 1$, we can no longer neglect spontaneous emission. One must then use equations (2.60) and (2.61). In the case of a four-level system, one obtains

$$\frac{dF}{dt} = -\frac{F}{\tau_{\text{cav}}} + \kappa \Delta N (F + 1) = \kappa [\Delta N (F + 1) - \Delta N_{\text{th}} F] .$$  (3.43)
3.3. LASER POWER

In steady-state regime, this leads to:

$$\frac{\Delta N}{\Delta N_{\text{th}}} = \frac{F}{F + 1}. \quad (3.44)$$

Besides, the saturation of the active medium reads

$$\frac{\Delta N}{\Delta N_0} = \frac{1}{1 + F/F_{\text{sat}}}. \quad (3.45)$$

By defining the excitation ratio as $r = \Delta N_0/\Delta N_{\text{th}}$ and combining equations (3.44) and (3.45), one obtains:

$$\frac{F}{F_{\text{sat}}} = \frac{r - 1}{2} + \sqrt{\left(\frac{r - 1}{2}\right)^2 + \frac{r}{F_{\text{sat}}}}. \quad (3.46)$$

What is the order of magnitude of $F_{\text{sat}}$? To answer that question, let us take the example of a Nd:YAG laser emitting a power $P = 100$ mW at $\lambda = 1.06 \mu\text{m}$. This corresponds to a photon flux $\Phi = \frac{P}{hc_0/\lambda} = 5.3 \times 10^{16} \text{ photons/s.}$

Let us imagine that the cavity is a 20-cm long two-mirror linear cavity, i.e., $L_{\text{cav}} = 0.4 \text{ m}$, and that the only noticeable cavity losses are the 2% transmission of the output coupler. The lifetime of the photons in that cavity is thus $\tau_{\text{cav}} = L_{\text{cav}}/c_0\Pi = 6.7 \times 10^{-8} \text{ s.}$ The number of photons in the cavity is hence $F = \tau_{\text{cav}} \Phi = 3.5 \times 10^{10}$. As one has, far above threshold, $F = F_{\text{sat}}(r - 1)$, the order of magnitude of $F_{\text{sat}}$ is $10^{10}$.

![Figure 3.7](image)

Figure 3.7: Number of photons $F$ in the cavity versus excitation ratio $r$ for $F_{\text{sat}} = 10^{10}$.

Figure 3.7 reproduces, in a semi-log scale, the evolution of the number of intracavity photons versus $r$ for $F_{\text{sat}} = 10^{10}$, plotted using equation (3.46).
The threshold corresponds to a dramatic increase of $F$, with a slope equal to $F_{\text{sat}}/2$.

**Comment:** The typical number ($10^{10} - 10^{12}$) of intracavity photons that we have obtained explains the remarkable properties of lasers. If one remembers that a “classical” source is unable to put 1 photon in a single mode of the electromagnetic field, it becomes clear that this explains the spatial and temporal coherence properties of laser light.

### 3.4 Spatial hole burning in a linear cavity

Up to now, we have always assumed that the lasing intracavity mode was a traveling plane wave. However, this is possible only in a unidirectional ring cavity. In a so-called “linear” cavity such as the one of figure 3.8, light makes round-trips and creates a standing wave. Contrary to the case of the traveling wave, the light energy density is no longer homogeneous along the propagation axis, leading to a drastic modification of the saturation of the active medium and of the behavior of the laser.

![Figure 3.8: Laser based on a linear cavity sustaining standing waves.](image)

#### 3.4.1 Standing wave. Saturation

If there were no interferences between the two intracavity counterpropagating traveling waves of intensities $I_+$ and $I_-$, the total intracavity intensity would be $I = I_+ + I_- \quad (\simeq 2I$ if the losses are small and $I_+ \simeq I_- \simeq I$) and the laser steady-state regime could be obtained by considering that this total intensity saturates the gain. But the interferences create a spatial modulation of the energy density $u_{\text{standing}}(z)$ along $z$, and thus the saturation of the population
3.4. SPATIAL HOLE BURNING

inversion depends on $z$ (this is the so-called spatial hole burning) according to:

$$\Delta n(z) = \frac{1}{\Delta n_0} \frac{1}{1 + \frac{u_{\text{standing}}(z)}{u_{\text{sat}}}},$$

(3.47)

with

$$u_{\text{standing}}(z) = 4u \sin^2 \left( \frac{n_0 \omega}{c_0} z \right),$$

(3.48)

and

$$u_{\text{sat}} = \frac{n_0}{c_0} I_{\text{sat}}.$$  

(3.49)

The energy density $u$ for one traveling wave is related to $I_+$ and $I_-$ like in equation (1.1).

3.4.2 Output power

One can derive the laser output power in the presence of spatial hole burning by adapting the following equation

$$\frac{dF}{dt} = -\frac{F}{\tau_{\text{cav}}} + \kappa F \Delta N$$

(3.50)

to the case of a laser exhibiting spatial hole burning. The variation due to the gain of the energy $W$ stored inside the cavity reads:

$$\left. \frac{dW}{dt} \right|_{\text{gain}} = \frac{c_0}{n_0} \sigma S \int_0^{L_a} \Delta n_0 u_{\text{standing}}(z) \frac{1}{1 + u_{\text{standing}}(z)/u_{\text{sat}}} \, dz.$$  

(3.51)

By using equation (3.48), we obtain, after integration along $z$: 

$$\left. \frac{dW}{dt} \right|_{\text{gain}} = \frac{c_0}{n_0} \sigma S \Delta n_0 u_{\text{sat}} L_a \left( 1 - \frac{1}{\sqrt{1 + 4u/u_{\text{sat}}}} \right).$$

(3.52)

In steady-state regime, this gain must exactly compensate the losses given by:

$$\left. \frac{dW}{dt} \right|_{\text{losses}} = \frac{2u V_{\text{cav}}}{\tau_{\text{cav}}},$$

(3.53)

leading to:

$$\frac{u}{u_{\text{sat}}} = \frac{r}{2} \left( 1 - \frac{1}{\sqrt{1 + 4u/u_{\text{sat}}}} \right).$$

(3.54)
This third order polynomial equation can be solved. By keeping only the physically meaningful solution we get:

\[ u = \frac{u_{\text{sat}}}{2} \left( r - \frac{1}{4} - \sqrt{\frac{r}{2} + \frac{1}{16}} \right). \]  

(3.55)

Finally, the laser output intensity is given by

\[ I_{\text{out}} = \frac{T}{2} I_{\text{sat}} \left( r - \frac{1}{4} - \sqrt{\frac{r}{2} + \frac{1}{16}} \right). \]  

(3.56)

This result must be compared with the expression of the laser output intensity in the absence of spatial hole burning:

\[ I_{\text{out}} = \frac{T}{2} I_{\text{sat}} (r - 1). \]  

(3.57)

Figure 3.9 shows that spatial hole burning, even if it does not modify the laser threshold, modifies the slope of its output power characteristics by a factor of 2/3. This is due to the fact that the standing wave does not optimally extract the energy out of the active medium.

### 3.4.3 Multimode operation

Let us consider for example an active medium which completely fills the cavity and let us consider two successive longitudinal modes of the cavity of wavelengths \( \lambda_q = \frac{L_{\text{cav, opt}}}{q} \) and \( \lambda_{q+1} = \frac{L_{\text{cav, opt}}}{(q + 1)} \). In the middle of
the cavity for example (see figure 3.10), the mode $\lambda_q$ does not saturate the atoms, while mode $\lambda_{q+1}$ efficiently saturates them. One can thus see that the gain competition for these two modes is strongly reduced, allowing them to oscillate simultaneously by using different spatial classes of atoms. In these conditions, multimode operation is favored by the fact that the different modes are amplified by different sets of active atoms.

![Figure 3.10: Standing wave structures of two successive longitudinal modes.](image)

In order to reduce the influence of spatial hole burning, one can use a rather short ($L_a \ll L_{cav}/2$) amplifying medium located at a short optical distance $a \ll L_{cav,\text{opt}}/2$ with respect to one of the cavity mirrors, i.e., where the successive longitudinal modes exhibit a large spatial overlap (see figure 3.10). Let us suppose that a first mode of wavelength $\lambda_1 = 2a/m_1$ is oscillating in the cavity. It exhibits a node in the center of the amplifying medium. Then, another cavity mode of wavelength $\lambda_2 = 2a/(m_2 + 1/2)$, spatially shifted by $\lambda/4$ in the active medium, will be almost unaffected by the saturation due to the first mode. Let us compare the frequency difference $\Delta \nu_p$ between these two “spatial hole burning modes” and the free spectral range of the cavity $\Delta \nu_{cav}$:

$$\frac{\Delta \nu_p}{\Delta \nu_{cav}} = \frac{\nu_2 - \nu_1}{c_0/L_{cav,\text{opt}}} = \frac{(m_2 - m_1 + 1/2)(c_0/2a)}{c_0/L_{cav,\text{opt}}} = \left(p + \frac{1}{2}\right) \frac{L_{cav,\text{opt}}}{a}.$$  

For $L_{cav,\text{opt}} \gg a$, the splitting between the spatial hole burning modes is much larger than the free spectral range of the cavity. The spatial shift between the different standing wave patterns in the active medium will thus be very small, in favor of a monomode operation of the laser.
3. STEADY-STATE REGIME
Chapter 4

Laser based on an inhomogeneously broadened medium

4.1 Inhomogeneously broadened medium

In the preceding chapters, we considered only homogeneously broadened gain media, i.e., gain media in which the $N$ atoms that interact with the field are identical (same transition frequencies, same linewidths, same selection rules with respect to the field,...). It is obvious that this constitutes a limit case which is only rarely observed in the real world. In contrast with this homogeneous broadening, there exists many cases in which the atoms, because of their behavior or their environment, exhibit slightly different resonance frequencies. The whole set of atoms is then said to lead to a inhomogeneously broadened transition, induced by the dispersion of the resonance frequencies of the individual atoms, each atom keeping the same homogeneously broadened profile. The broadening is said to be “purely inhomogeneous” if the inhomogeneous profile is much broader than the homogeneous profile. In general, the two types of broadening coexist.

4.1.1 Gaussian profile. Doppler effect. Ions embedded in a crystalline matrix

The statistical origin of the inhomogeneous broadening often (but not always!) leads to a Gaussian profile. If $\Delta \nu_{\text{inhom}}$ is the full width at half maxi-
mum, the normalized profile reads:

\[ G(\nu - \nu_0) = \frac{2}{\Delta \nu_{\text{inhom}}} \sqrt{\frac{\ln 2}{\pi}} \exp \left[ -\ln 2 \left( \frac{\nu - \nu_0}{\Delta \nu_{\text{inhom}}/2} \right)^2 \right]. \] (4.1)

The Doppler broadening is one example of such a Gaussian inhomogeneous broadening which occurs in gases. Let us consider an atom moving at velocity \( \mathbf{v} \) with respect to the observer’s frame. \( v_z \) is the projection of \( \mathbf{v} \) along the observation axis \( Oz \). The wave emitted by these atoms at resonance frequency is detected by the observer with a frequency shift given, at first order in \( v_z/c \), by:

\[ \frac{\nu - \nu_0}{\nu_0} = \frac{v_z}{c_0}. \] (4.2)

In a gas at thermal equilibrium at temperature \( T \), the components \( v_z \) of the velocity obey the Maxwell probability distribution law:

\[ P(v_z)dv_z = \left( \frac{m}{2\pi k_B T} \right)^{1/2} \exp \left[ -\frac{mv_z^2}{2k_B T} \right] dv_z, \] (4.3)

where \( k_B \) is Boltzmann’s constant.

The intensity emitted along \( Oz \) by the class of atoms whose measured frequencies lie in the interval \( [\nu, \nu + d\nu] \) is proportional to this probability in which \( v_z \) is replaced by its value given by (4.2). The normalized profile and the “Doppler” FWHM \( \Delta \nu_D \) are then given by

\[ G_D(\nu - \nu_0) = \frac{c_0}{\nu_0} \sqrt{\frac{m}{2\pi k_B T}} \exp \left[ -\frac{mc_0^2}{2k_B T} \left( \frac{\nu - \nu_0}{\nu_0} \right)^2 \right], \] (4.4)

\[ \Delta \nu_D = \frac{\nu_0}{c_0} \sqrt{\frac{8RT \ln 2}{M}}, \] (4.5)

where \( M \) is the molar mass of the gas and \( R \) is the ideal gas constant. In practice, at room temperature, the Doppler widths are typically of the order of 1 GHz.

The active ions embedded in a crystalline or amorphous matrix are another example of inhomogeneous broadening. Because of the presence of small defects on the crystal, the different crystalline sites in which the dopant ions sit are not exactly identical, even if the local symmetry is preserved. The crystal field created by the environing ions varies from one site to the other, making vary the energies of the levels of the doping ion. If the energy distribution is Gaussian, then the profile of the resonance lines is also Gaussian.
A third example is given by the existence of several isotopes of the active
atom. Indeed, the transition frequencies of the different isotopes are slightly
different, leading to an inhomogeneous broadening.

4.1.2 Unsaturated amplification coefficient

In the general case, let us call \( P(\nu - \nu_0) \) the distribution, centered in \( \nu_0 \), of
the resonance frequencies \( \nu_i \) of the atoms [see figure 4.1(a)]. Then, if \( \Delta n_0 \)
is the total unsaturated population inversion (per unit volume) and if we
suppose that the efficiency of the pumping is the same for all the atoms,
independently of their central frequency \( \nu_i \), then the population difference
\( d\Delta n_i \) for the atoms whose resonance frequency lies in the interval \([\nu_i, \nu_i + d\nu_i]\)
reads:

\[
d\Delta n_i = \Delta n_0 P(\nu - \nu_0) d\nu_i .
\]  

\[
(4.6)
\]

Figure 4.1: Unsaturated gain for an inhomogeneously broadened medium.
(a) Distribution \( P \) of the resonance frequencies \( \nu_i \) of the atoms. If the pump-
ing has the same efficiency for all the atoms, this profile is proportional to the
spectral distribution of the population inversion. (b) Homogeneous profile
of the gain for atoms whose resonance frequency is \( \nu_i \). (c) The unsaturated
gain \( \alpha_0 \) is the result of the convolution of the profile represented in (a) and
(b).

For this class of atoms, the cross section \( \sigma_i(\nu) \) is proportional to the
normalized profile \( g(\nu - \nu_i) \) centered at frequency \( \nu_i \) [see figure 4.1(b)]:

\[
\sigma_i(\nu) = \frac{\sigma_0}{g(0)} g(\nu - \nu_i) .
\]  

\[
(4.7)
\]

The unsaturated amplification coefficient \( \alpha_0(\nu) \) is calculated by summing
over all the classes of atoms:

\[
\alpha_0(\nu) = \frac{\sigma_0}{g(0)} \Delta n_0 \int_{-\infty}^{\infty} g(\nu - \nu_i) P(\nu_i - \nu_0) d\nu_i .
\]  

\[
(4.8)
\]
This integral, which is the convolution of two normalized functions, is also a normalized function. The result of this convolution is schematized in figure 4.1(c). In the case where the homogeneous profile is Lorentzian (g ≡ \(L\) with width \(\Delta \nu_{\text{hom}}\)) and where the inhomogeneous profile is Gaussian (\(P \equiv G\) with width \(\Delta \nu_{\text{inhom}}\)), the result of the convolution is called a Voigt profile.

The application of (4.8) to the limiting cases of pure homogeneous and pure inhomogeneous profiles leads to:

\[
\begin{align*}
\alpha_0(\nu) &= \sigma(\nu) \Delta n_0 \quad \text{if} \quad \Delta \nu_{\text{inhom}} \ll \Delta \nu_{\text{hom}}, \\
\alpha_0(\nu) &= \frac{\sigma_0}{g(0)} P(\nu - \nu_0) \Delta n_0 \quad \text{if} \quad \Delta \nu_{\text{hom}} \ll \Delta \nu_{\text{inhom}},
\end{align*}
\]

(4.9) (4.10)

In the case of optical transitions in gases, the Doppler width can be two orders of magnitude larger than the natural linewidth, leading to Gaussian line profiles.

4.1.3 Spectral hole burning

Let us now wonder what happens when the population difference is saturated by a wave oscillating at frequency \(\nu\). By taking into account the fact that the saturation intensity \(I_{\text{sat},i}(\nu)\) for atoms resonant at frequency \(\nu_i\) saturated by a wave at frequency \(\nu\) evolves as the inverse of the cross section, we have, using (4.7):

\[
I_{\text{sat},i}(\nu) = \frac{h \nu}{2^* \sigma_i(\nu) \tau} = I_{\text{sat}0} \frac{g(0)}{g(\nu - \nu_i)},
\]

(4.11)

where \(I_{\text{sat}0} = h \nu/2^* \sigma_0 \tau\) is the saturation intensity at resonance and where \(2^*\) is equal to 1 or 2 for a 4- or 3-level system respectively. Using (4.11), the population difference for atoms resonant at \(\nu_i\) reads:

\[
\begin{align*}
\mathrm{d}\Delta n_i &= P(\nu_i - \nu_0) \Delta n_0 \frac{1}{1 + \frac{I}{I_{\text{sat},i}(\nu)}} \mathrm{d}\nu_i \\
&= P(\nu_i - \nu_0) \Delta n_0 \frac{1}{1 + \frac{I}{I_{\text{sat}0}} \frac{g(\nu - \nu_i)}{g(0)}} \mathrm{d}\nu_i \\
&= P(\nu_i - \nu_0) \Delta n_0 \left\{ 1 - \frac{I}{I_{\text{sat}0}} \frac{g(0)}{g(\nu - \nu_i)} \right\} \mathrm{d}\nu_i.
\end{align*}
\]

(4.12)

The term \(P(\nu_i - \nu_0) \Delta n_0\) is the “unsaturated spectral population difference” of figure 4.1, which exhibits the initial inhomogeneous profile. The second
4.1. INHOMOGENEOUSLY BROADENED MEDIUM

term between the braces takes non-zero values only when the class of atoms is resonant with the laser ($\nu \simeq \nu_i$). We thus burn a “hole” centered at the laser frequency in the population difference spectral profile. Its relative depth is $\frac{I}{I_{sat0}}$, this is the so-called “spectral hole burning” phenomenon described in figure 4.2(a).

Figure 4.2: Saturated gain for an inhomogeneously broadened medium. (a) Spectral distribution of the unsaturated population inversion (dashed line) and of the population inversion saturated by a laser at frequency $\nu$ (full line). (b) Unsaturated gain $\alpha_0$ and saturated gain $\alpha$ versus laser frequency.

In the case of a Lorentzian homogeneous profile one has

$$g(\nu - \nu_i) \equiv L(\nu - \nu_i) = \frac{2}{\pi \Delta \nu_{hom}} \frac{1}{1 + \left[2(\nu - \nu_i)/\Delta \nu_{hom}\right]^2},$$

leading to the following expression for the second term between the braces of equation (4.12), which describes the hole:

$$\frac{I}{I_{sat0}} + \frac{g(0)}{g(\nu - \nu_i)} = \frac{I}{I_{sat0}} \frac{1}{1 + \frac{I}{I_{sat0}} \left(\frac{2(\nu - \nu_i)}{\Delta \nu_{hom} \sqrt{1 + I/I_{sat0}}}\right)^2}.$$  

Consequently, in the approximation where $P(\nu_i - \nu_0) \simeq P(\nu - \nu_0)$, the hole has a Lorentzian profile with a width given by:

$$\Delta \nu' = \Delta \nu_{hom} \sqrt{1 + \frac{I}{I_{sat0}}}.$$  

In conclusion, only the atomic classes whose resonance frequencies are very close to the laser frequency efficiently interact with the laser. The other atoms are just passive spectators.
4.1.4 Saturated amplification coefficient

Let us now calculate the saturated amplification coefficient with the method used in (4.8) and the result (4.12). The resulting expression can be cast in the following form:

\[ \alpha(\nu) = \sigma_0 \Delta n_0 \int_{-\infty}^{\infty} d\nu_i P(\nu_i - \nu_0) \frac{1}{g(0)} \frac{g(\nu - \nu_i)}{g(\nu_i - \nu)} + \frac{I}{I_{sat0}} . \tag{4.16} \]

Even if \( g \equiv L \) and \( P \equiv G \), this integration can usually not be performed analytically. However, it leads to an analytical result for a purely inhomogeneous broadening for which the profile \( P(\nu_i - \nu_0) \simeq P(\nu - \nu_0) \) can be taken out of the integral. Then, by performing the integration and applying (4.10), one obtains a remarkably simple relation between \( \alpha(\nu) \) and \( \alpha_0(\nu) \):

\[ \alpha(\nu) = \alpha_0(\nu) \sqrt{1 + \frac{I}{I_{sat0}}} = \alpha_0(\nu_0) \sqrt{1 + \frac{I}{I_{sat0}}} \frac{P(\nu - \nu_0)}{P(0)} . \tag{4.17} \]

This result is represented in figure 4.2(b). One can see that, finally, saturation does not alter the shape of the inhomogeneous profile, but reduces the amplification coefficient by a factor \( \sqrt{1 + \frac{I}{I_{sat0}}} \). This result applies because the frequency of the wave that probes the gain is the same as the one which saturates this gain. This is completely different from a “pump-probe” experiment performed with two waves of different frequencies.

4.2 Laser operation

4.2.1 Application of the oscillation condition. Multi-mode operation

Let us consider a laser built from an active medium whose broadening is mainly inhomogeneous \((\Delta \nu_{hom} \ll \Delta \nu_{inhom})\). The longitudinal modes of frequencies \( \nu_q \) such that \( \alpha_0(\nu_q) > \Pi/L_\alpha \) may then oscillate simultaneously. Indeed, we have seen in section 4.1 that, if \( \Delta \nu_{hom} \ll c_0/L_{cav,opt} \), each frequency \( \nu_q \) corresponds to a particular class of atoms which interact only with that mode. The light in these modes thus burns holes in the unsaturated gain profile \( \alpha_0(\nu) \) [see equation (4.10)]. If \( \Delta \nu_{hom} \ll c_0/L_{cav,opt} \), these holes can be considered as completely independent, and thus the different longitudinal modes interact with separate sets of atoms. Let us call \( I_q \) the intensity of
Multimode oscillation

Figure 4.3: Unsaturated (dashed line) and saturated (full line) gain versus frequency. We notice the presence of holes at the frequencies of the oscillating modes, thus ensuring the equality of gain and losses at each of these frequencies.

mode \( q \). If the homogeneous profile of the transition has a Lorentzian shape, we can then apply equation (4.17) to write the equality of gain and losses:

\[
\alpha(\nu_q) = \frac{\alpha_0(\nu_q)}{\sqrt{1 + I_q/I_{sat0}}} = \frac{\Pi}{L_a}, \tag{4.18}
\]

leading to the expression of \( I_q \):

\[
I_q = I_{sat0} \left[ \left( \frac{\alpha_0(\nu_0) L_a}{\Pi} \frac{P(\nu_q - \nu_0)}{P(0)} \right)^2 - 1 \right]. \tag{4.19}
\]

The intensity of the mode varies linearly with the square of the ratio \( \alpha_0(\nu_0)L_a/\Pi \), it is maximum when \( \nu_q = \nu_0 \), minimum at the limit of the oscillation domain \( \delta \nu_{osc} \) defined by the condition \( \alpha_0(\nu_q) > \Pi/L_a \). The number of oscillating modes is given by the ratio \( L_{cav,opt} \delta \nu_{osc}/c_0 \), to within \( \pm 1 \). For example, a He-Ne laser with a linear cavity \( (L_{cav,opt}/2 = 50\text{cm}) \) with a Doppler oscillation range \( \delta \nu = 1500 \text{ MHz} \) can oscillate on 5 modes.
4.2.2 Output power of a given mode

The calculation is formally identical to the one of subsection 3.3.1, using the same notations. The output intensity of mode $q$ is thus given by:

$$I_{\text{out}} = TI_{\text{sat}0} \left[ \frac{\alpha_0(\nu_q) L_a}{T + T} \right]^2 - 1.$$  \hspace{1cm} (4.20)

The optimal output coupling is obtained as in subsection 3.3.1. The result is less simple than (3.40), because it involves a third order equation, but the shape of the curve is roughly the same. In the case of strong pumping ($\alpha_0(\nu_q) L_a \gg T$), one can show that the optimal coupling value $T_{\text{opt}}$ is approximately equal to $T$.

4.2.3 Mode competition: beyond the “rate equations” model

The cw operation of a multimode laser leads to several new problems:

1. Mode competition: will the modes oscillate simultaneously or will one of them oscillate alone (see part II)?

2. Mode intensities: is equation (4.20) really valid in the case of inhomogeneous broadening, and what happens for a homogeneous broadening?

3. The relative phase of the electric fields of the modes: this is the general problem of laser “mode locking” (see part II).

4.3 Special case of gas lasers

4.3.1 Spectral hole burning in a linear cavity

Let us consider the case of a monomode gas laser based on a cavity sustaining a standing wave: such a wave is the superposition of two traveling waves propagating in opposite directions. They are resonant with two classes of atoms of longitudinal velocities $v_+$ and $v_-$ given by:

$$\nu \left( 1 - \frac{v_+}{c_0} \right) = \nu_0 ,$$ \hspace{1cm} (4.21)

$$\nu \left( 1 + \frac{v_-}{c_0} \right) = \nu_0 ,$$ \hspace{1cm} (4.22)
leading to
\[ v_+ = -v_- = \frac{c_0}{\nu} (\nu - \nu_0) . \] (4.23)

The distribution of \( \Delta n \) versus longitudinal velocity \( \nu \) thus has two holes located symmetrically with respect to \( \nu = 0 \). The laser thus extracts energy coming from two different velocity classes. For \( \nu = \nu_0 \), these two holes merge into a single hole at the middle of the Doppler distribution: the atoms of longitudinal velocity \( \nu = 0 \) interact with the two waves, thus experiencing a double saturation. If we neglect spatial hole burning, we can write the gain as [see (4.17)]:
\[ \nu \neq \nu_0 \Rightarrow \alpha(\nu) = \frac{\alpha_0(\nu)}{\sqrt{1 + I/I_{sat0}}} , \] (4.24)
\[ \nu = \nu_0 \Rightarrow \alpha(\nu) = \frac{\alpha_0(\nu)}{\sqrt{1 + 2I/I_{sat0}}} . \] (4.25)

When \( \nu \to \nu_0 \), the intensity of each of the traveling waves decreases by a factor of 2, and so does the power of the output beam. This decrease occurs over a spectral width more or less equal to the homogeneous linewidth, where the two holes have a strong overlap. This power dip, called the “Lamb dip”, has been theoretically predicted by Willis E. Lamb, Jr. [Physical Review 134, 1429 (1964)] and experimentally observed in many gas lasers. Let us study it in more details.

### 4.3.2 Lamb dip

We can study this phenomenon in an approximated manner using equation (4.16) in which we include the return wave in the saturation term; if \( \nu_i \) is the frequency of the wave propagating in the \(+z\) direction and resonant with the atomic class of longitudinal velocity \( \nu \), the frequency of a wave propagating along the \(-\nu\) direction and resonant with the same class of atoms is \( \nu_i' = 2\nu_0 - \nu_i \). Indeed:
\[
\begin{align*}
\nu_i \left( 1 - \frac{\nu_0}{c_0} \right) &= \nu_0 \\
\nu_i' \left( 1 + \frac{\nu_0}{c_0} \right) &= \nu_0
\end{align*}
\Rightarrow \nu_i + \nu_i' = \frac{2\nu_0}{1 - (\nu/c_0)^2} \approx 2\nu_0 \] (4.26)

In these conditions, equation (4.16) must be rewritten:
\[
\alpha(\nu) = \sigma_0 \Delta n_0 \int_{-\infty}^{\infty} G(\nu_i - \nu_0) \left[ \frac{I}{I_{sat0}} + \frac{L(0)}{L(\nu - \nu_i)} + \frac{I}{I_{sat0}} \frac{L(\nu - 2\nu_0 + \nu_i)}{L(\nu - \nu_i)} \right]^{-1} d\nu_i ,
\] (4.27)
where $G(\nu_i - \nu_0)$ is the Gaussian Doppler profile, and $L(\nu - \nu_i)$ is the Lorentzian homogeneous profile of a transition centered at frequency $\nu_i$. This frequency $\nu_i$ is here the integration variable.

The integration in (4.27) must usually be performed numerically. However, similarly to subsection 4.1.4, it can be calculated analytically in the case of a purely inhomogeneous broadening in which the profile $G(\nu_i - \nu_0)$ gets out of the integral. By performing this integration, one finds that the FWHM of the Lamb dip is close to the homogeneous width.

### 4.3.3 Reverse Lamb dip

![Figure 4.4: Intensity versus frequency for a gas laser containing a saturable absorber.](image)

A similar phenomenon is known as the “reverse Lamb dip” observed in a laser containing an intracavity saturable absorber cell (see figure 4.4). When the mode frequency coincides with the resonance frequency $\nu_{\text{abs}}$ of the absorbing gas, this latter becomes more transparent: one observes a power peak superimposed on the power versus frequency profile of the laser. This phenomenon is routinely used to stabilize the frequency of gas lasers.

### 4.3.4 Spatial hole burning.

A more precise analysis of the Lamb dip phenomenon must take the “spatial hole burning” into account (see section 3.4). The integration in equation
(4.27) must be completed by an integration on \( z \) along the standing wave structure in the active medium.

### 4.4 Mode selection

#### 4.4.1 Different techniques to make the laser monomode

Most of the times, one wishes cw lasers to be spectrally as narrow as possible, and, if possible, a single-frequency laser operation.

Figure 4.5 reproduces the scheme of a dye laser pumped by an argon ion laser: it illustrates classical mode selection techniques.

![Figure 4.5: Example of a commercial cw dye laser based on a ring cavity containing étalons, filters, and an optical diode, in order to reach a tunable single-frequency oscillation regime (Taken from A. E. Siegman, Lasers, op. cit.).](image)

If the amplifying medium (a dye jet with a thickness smaller than 1 mm) is alone in the cavity, the laser operated in the two propagation directions in an unstable manner. The optical diode ensures a stable unidirectional operation, but based on \( 10^4 \) to \( 10^5 \) longitudinal modes (linewidth of the order of \( 10^{13} \) Hz).

A first selection is performed using a birefringent filter (so-called Lyot filter) under Brewster incidence, which plays the role of a dispersive element with a poor resolution, quite analogous to a prism or a grating. In gas lasers (He-Ne, argon ion,...) a prism permits to select a given transition among discrete lines. In pulsed laser, a grating replacing one of the cavity mirrors efficiently selects a part of the spectrum.
A second selection is performed by a thin Fabry-Perot étalon (few mm thickness, finesse \(\simeq 10\)), and a third by a thick Fabry-Perot étalon (few cm thick, finesse \(> 10\)). Thanks to these étalons, only one longitudinal mode fulfills the oscillation condition \(a_0(\nu_q) > \pi/L_a\). The continuous tuning of the mode frequency is obtained by translating one of the cavity mirrors using a piezo-electric transducer. An electronic servo-loop ensures locks the laser to its maximum intensity and ensures that the transmission peaks of the intracavity étalons follow the lasing mode during these frequency sweeps. Different technologies (angle tuning, temperature tuning, pressure tuning, use of piezo-electric transducers,...) are used to adjust the spectral position of the étalons.

A very useful method to select one mode consists in adding extra mirrors to the cavity, in order to create extra intracavity interferometers (Michelson, Sagnac, Mach-Zehnder, Fox-Smith configurations,...). The properties of three-mirror cavities, depending on the chosen configuration, are often difficult to study and often require the servo-locking of one cavity on the other [see P. W. Smith, "Mode selection in lasers", Proc. IEEE 60, 422 (1972)].

### 4.4.2 Laser frequency stabilization

The selection of one laser modes does not ensure its frequency stability. In the preceding example of a dye laser, a small part of the output power can for example be isolated and sent to an external reference high finesse, ultra-stable, temperature stabilized Fabry-Perot cavity. The laser frequency is then locked to one resonance of the Fabry-Perot cavity. This technique can of course be applied to other types of interferometric references (Michelson interferometer,...).

A much better stability and a better reproducibility can be obtained by locking the laser to an absorption line of an atomic or molecular gas and by getting rid of the Doppler broadening using so-called sub-Doppler spectroscopy techniques: use of the fluorescence of a transversally excited beam, saturated absorption, etc...
5.1 Transient laser behavior

Like in any physical system, the laser reaches its steady-state regime after having experienced a transient regime which may be oscillatory or not. Moreover, even if the laser is driven by stable drivers or pumped by a very quiet pump, there still exists tiny external perturbations (mechanical or acoustic vibrations, thermal noise, electric noise,...) which excite the system and move it away from its steady-state operation point. We shall see that the manner (quasi-periodic or overdamped) in which the laser goes back to its steady-state regime depends on the dynamical class (class-A or class-B, see subsection 2.2.2) of the laser, i. e., of the relative values of $\tau$ and $\tau_{\text{cav}}$. We shall also see that this determines the turn-on behavior of the laser and we shall see in chapter 6 that this has also important consequences on the laser noise.

5.1.1 Description in single-frequency regime: relaxation oscillations

We restrict the present discussion to homogeneously broadened gain media, and we suppose that the cavity losses are small and the laser oscillates in single-frequency regime. In these conditions, equations (2.59) and (2.60)
describe the laser behavior:

\[
\frac{dF}{dt} = -\frac{F}{\tau_{\text{cav}}} + \kappa F \Delta N, \tag{5.1}
\]

\[
\frac{d\Delta N}{dt} = -\frac{1}{\tau} (\Delta N - \Delta N_0) - 2^* \kappa F \Delta N. \tag{5.2}
\]

Moreover, we have seen in subsection 3.1.2 that, on its way back to its steady-state regime, the laser behavior is governed by the following Lyapunov exponents:

\[
\lambda_{\pm} = -\frac{r}{2\tau} \pm \frac{1}{2} \sqrt{\left(\frac{r}{\tau}\right)^2 - \frac{4(r - 1)}{\tau \tau_{\text{cav}}}}. \tag{5.3}
\]

The general solution describing the laser evolution towards the steady-state solution thus reads:

\[
F(t) = F_{\text{ON}} + \delta F_+ e^{\lambda_+ t} + \delta F_- e^{\lambda_- t}, \tag{5.4}
\]

\[
\Delta N(t) = \Delta N_{\text{ON}} + \delta N_+ e^{\lambda_+ t} + \delta N_- e^{\lambda_- t}, \tag{5.5}
\]

where \(\delta F_+, \delta F_-, \delta N_+ \text{ et } \delta N_-\) depend on the initial conditions. We can thus see that the laser behavior depends strongly on the sign of the expression below the square root in equation (5.3).

### 5.1.1.1 Case of a class-A laser: \(\tau_{\text{cav}} \gg \tau\)

In the case of a class-A laser, the lifetime of the population inversion is much shorter than the lifetime of the photons in the cavity ("good cavity limit") leading to:

\[
\lambda_+ \simeq -\frac{r - 1}{r} \frac{1}{\tau_{\text{cav}}}, \tag{5.6}
\]

\[
\lambda_- \simeq -\frac{r}{\tau}. \tag{5.7}
\]

One can thus see that \(\lambda_- \gg \lambda_+\). The laser will thus return exponentially to its steady-state regime with a relaxation time:

\[
\tau_{\text{relax}} = \frac{r}{r - 1} \tau_{\text{cav}} \simeq \tau_{\text{cav}} \text{ as soon as } r \gg 1. \tag{5.8}
\]

We could have obtained this result by noticing that the condition \(\tau_{\text{cav}} \gg \tau\) allows us to adiabatically eliminate the population inversion in equations (5.1) and (5.2). Indeed, by writing \(d\Delta N/dt = 0\) in equation (5.2), we get:

\[
\Delta N = \frac{\Delta N_0}{1 + 2^* \kappa F}. \tag{5.9}
\]
5.1. LASER TRANSIENT

which can be injected into equation (5.1) to lead to:

\[
\frac{dF}{dt} = F \left( -\frac{1}{\tau_{\text{cav}}} + \frac{\kappa \Delta N_0}{1 + 2\kappa \tau F} \right). \tag{5.10}
\]

This equation clearly shows that in the case of class-A dynamical regime, the instantaneous saturation of the gain is responsible for the fact that the laser is brought back to its equilibrium position in a monotonous manner. Indeed, if \( F < F_{\text{ON}} \), the gain term \( \frac{\kappa \Delta N_0}{1 + 2\kappa \tau F} \) becomes larger than the loss term \( \frac{1}{\tau_{\text{cav}}} \), allowing the laser intensity to increase. Conversely, when \( F > F_{\text{ON}} \), the gain term \( \frac{\kappa \Delta N_0}{1 + 2\kappa \tau F} \) becomes smaller than the loss term \( \frac{1}{\tau_{\text{cav}}} \) leading to a decrease of the intensity.

5.1.1.2 Case of the class-B laser: \( \tau \sim \tau_{\text{cav}} \) or \( \tau > \tau_{\text{cav}} \)

In this case the transient regime is quasi-periodic with a frequency and a damping time given by:

\[
\begin{align*}
\tau_{\text{relax}} &= \frac{1}{2\pi} \sqrt{\frac{r - 1}{\tau_{\text{cav}}} - \left( \frac{r}{2\tau} \right)^2}, \\
\tau_{\text{damp}} &= \frac{2\tau}{r}.
\end{align*}
\tag{5.11, 5.12}
\]

Figure 5.1: Example of behavior of the number of photons in the cavity when the system is driven out of its equilibrium position. We have taken \( \tau = 240 \mu s, \tau_{\text{cav}} = 33\text{ns}, \) and \( r = 5 \).
5. TRANSIENT AND Q-SWITCHED OPERATIONS

In the (relatively common) case where \( \tau \gg \tau_{\text{cav}} \) ("bad cavity limit"), the frequency of these relaxation oscillations becomes:

\[
f_{\text{relax}} \simeq \frac{1}{2\pi} \sqrt{\frac{r-1}{\tau \tau_{\text{cav}}}}.
\]  

(5.13)

An example of these relaxation oscillations is presented in figure 5.1. This figure has been computed for the case of a Nd:YAG laser (4 levels, \( \tau = 240 \mu s \)) with a 50-cm long cavity with 5% losses (\( \tau_{\text{cav}} = 33 \text{ ns} \)) and operating 5 times above threshold. The plot represents the evolution of the number of photons \( F \) normalized to its steady-state value when one imposes, at \( t = 0 \), a number of photons equal to twice its steady-state value.

5.1.2 Switching on the laser: spiking

These differences between class-A and class-B lasers are even more striking when one looks at the laser behavior at laser turn-on. In this case, equations (5.1) and (5.2) can no longer be linearized around the steady-state solutions but must be numerically solved.

5.1.2.1 Case of the class-A laser: \( \tau_{\text{cav}} \gg \tau \)

Figure 5.2 reproduces the evolution of the population inversion and of the number of photons normalized to their steady-state values in the case of a class-A laser, obtained by numerically integrating equations (5.1) and (5.2). One can see that after a very short phase during which the population inversion increases to reach \( \Delta N_0 \), the laser seems to take a time of the order of 600 ns to start (see figure 5.2(b)). Actually, one can see in figure 5.2(c), which reproduces the same evolution of the number of photons on a log scale, that the laser starts as soon as the gain gets larger than the losses: the intensity then increases exponentially as long as \( F \) is small compared with \( F_{\text{sat}} \). Once the number of photons becomes significant, the population inversion instantaneously saturates and the system reaches its steady-state regime following a monotonous evolution.

5.1.2.2 Case of the class-B laser: \( \tau \sim \tau_{\text{cav}} \) or \( \tau > \tau_{\text{cav}} \)

Figure 5.3 reproduces the evolution of the population inversion and of the number of photons normalized to their steady-state values in the case of a class-B laser having the same characteristics as the one of figure 5.1. We can see that the damped oscillatory behavior of the laser becomes more spectacular here: at switch on, the laser emits a series of very intense pulses
5.1. LASER TRANSIENT

Figure 5.2: Behavior of a class-A laser after being switched on. We have taken $\tau = 10$ ns, $\tau_{\text{cav}} = 170$ ns, and $r = 5$.

(“spiking”) which slowly damp. Figure 5.1(c) reproduces the laser trajectory in the $(\Delta N, F)$ plane and permits to better understand the origin of this phenomenon: the reaction of the population inversion to the variations of the intra-cavity intensity is so slow that it is in phase quadrature with respect to $F$. This comes from equation (5.1), which shows that when the number of photons reaches its maximum value, i.e., when $dF/dt = 0$, $\Delta N$ is equal to its steady-state value $\Delta N_{\text{th}}$. 
5.2 Transient operation in multimode regime

In multimode regime, the theory is much more complicated because one cannot just consider the intensities of the oscillating modes as independent and add them. In reality, in order to take into account the spatial and temporal interferences of the modes, one must write one equation of evolution for each mode electric field. The system thus exhibits as many eigenfrequencies as the number of modes and the behavior becomes much more complicated and sometimes even chaotic as seen in figure 5.4 which has been obtained with a multimode ruby laser (horizontal scale: 10 µs/division).
5.3 Q-switched laser

5.3.1 Principe

We have seen in section 5.1 that when it is switched on, a class-B laser emits pulses whose peak intensity is much larger than its steady-state intensity. For example, by looking at figure 5.3, one can see that the first pulse has a peak intensity 20 times larger than the steady-state intensity. The aim of the laser “Q-switching” is to amplify this effect and to control the emission time of the pulse. One thus tries to oblige the laser to store in the form of population inversion in the active medium the energy provided by the pump during a time of the order of \( \tau \), and then to release this energy in a light pulse lasting a duration of the order of \( \tau_{\text{cav}} \). The peak intensity of the laser is then roughly equal to \( \tau/\tau_{\text{cav}} \) times the steady-state intensity that would be provided by the same laser in cw regime. It is worth noticing that this technique works only for class-B lasers. More precisely, it is all the more efficient for lasers for which \( \tau \gg \tau_{\text{cav}} \), permitting to reach peak intensities much stronger (typically \( 10^5 \) times stronger) than the cw intensity emitted by the same laser, but, of course, in a typically 10 ns long pulse.

In order to reach this goal, the “trick” consists in deliberately introducing strong optical losses inside the cavity (corresponding to a poor cavity quality factor \( Q_{\text{cav}} \)) during a time of the order of \( \tau \), in order to reach a large population inversion \( \Delta N \) without letting the laser reach threshold, i.e., without any saturation (see figure 5.5). When \( \Delta N \) reaches an initial value \( \Delta N_i \) close to the asymptotic value \( \Delta N_0 \) provided by the pumping strength and large compared to the threshold population inversion (without the extra losses) \( \Delta N_{\text{th}} \), one suddenly (in a time of the order of 1 ns) decreases the cavity losses: \( Q_{\text{cav}} \) increases very fast, explaining why this technique is known as “Q-switching”. Since \( \Delta N_i \gg \Delta N_{\text{th}} \), the number \( F \) of photons in the cavity increases very quickly. As seen above in the case of relaxation oscillations, it is maximum for \( \Delta N = \Delta N_{\text{th}} \).

The energy accumulated during the pumping duration (sometimes of the order of one millisecond) is suddenly released as a very short (10 ns) and very
intense (MW-GW peak power) optical pulse. After the pulse, $\Delta N$ decreases to a final value $\Delta N_f$ which is low, and even ideally zero for a well optimized four-level system. Before deriving the pulse parameters, let us discuss the Q-switching techniques.

5.3.2 Q-switching techniques

Two main kinds of Q-switching techniques must be distinguished (see figure 5.6): active (a,b,c) and passive (d) techniques.

a) Rotating mirror: this is the first method ever used (ruby laser); the mirror is mounted on a mount rotating at a fast angular velocity thanks to a pressurized air turbine. The mirror rotation and the pump flash must be carefully synchronized. This technique, which lacks reproducibility, is too noisy and difficult to align. It is no longer used.

b) Electro-optic cell (Pockels cell or liquid Kerr cell): the cavity contains a polarizer and an electro-optic element (sometimes coupled to a quarter-wave plate). During the pumping phase, it acts as a quarter-wave plate: the light reflected on one of the mirrors is rejected by the polarizer because the round-trip through the quarter-wave plate has rotated its polarization by $90^\circ$. To Q-switch the laser, the voltage applied on the electro-optic element is suddenly switched on or off (depending on the details of the setup) in order to allow light to freely make round-trips through the cavity without having its polarization

---

Figure 5.5: Principle of the laser Q-switching (Taken from A. E. Siegman, Lasers, op. cit.).
modified. This active Q-switching technique, which is one of the most often used, presents the advantages of being electrically controlled and to lead to a very fast switching.

c) **Acousto-optic modulator**: This intra-cavity element is spatially modulated by an acoustic wave. This modulation creates a Bragg grating which diffracts and deviates light outside the cavity in order to guarantee a low value for $Q_{\text{cav}}$. Stopping the modulation permits to trigger the laser pulse.

d) **Saturable absorber**: one introduces inside the cavity an absorber that can be easily saturated, i. e., that becomes transparent when it is shined with a relatively important intensity. If the saturation intensity of that absorber is low compared to the saturation intensity of the amplifying medium, then the losses of the cavity, which are initially strong, will decrease faster than the gain during the laser oscillation build-up, leading to a short and intense pulse. This type of passive Q-switching
is simple, efficient, and does not need any complicated electro-optic and
electronic elements. However, it suffers from fluctuations in the time
of emission of the pulses.

5.3.3 Theory of active Q-switching

Figure 5.5 presents the different phases of the active Q-switching process.
Provided a few approximations are performed, equations (5.1) and (5.2) per-
mit to analytically derive the laser parameters during the four main phases
of the laser Q-switching. We choose here a 4-level system \( (2^* = 1) \). How-
ever, the following discussion remains essentially valid in the case of a 3-level
system.

5.3.3.1 Pumping

Let us apply equation (5.2) with \( F = 0 \), allowing us to obtain the evolution
of \( \Delta N(t) \) provided the pumping \( \Delta N_0 \) is supposed constant during pumping:

\[
\Delta N(t) = \Delta N_0 \left(1 - e^{-t/\tau}\right).
\]  (5.14)

In order to optimize the laser efficiency, the pumping time is chosen equal to
3 or 4 times \( \tau \) in order to allow \( \Delta N_i \) to be very close to \( \Delta N_0 \). In the case of
solid-state lasers, this corresponds to pumping times typically of the order of
a few ms.

5.3.3.2 Beginning of the pulse

The pulse is built from spontaneous emission. We must consequently use
equation (2.61) with \( N_1 = 0 \) and \( \Delta N(t) = N_2 \). We suppose that \( \Delta N(t) \) does
not significantly vary during that phase: \( \Delta N(t) \simeq \Delta N_i \). We define

\[
r = \frac{\Delta N_i}{\Delta N_{th}},
\]  (5.15)

leading to:

\[
\frac{dF}{dt} = \frac{1}{\tau_{\text{cav}}} \left[r(F + 1) - F\right],
\]  (5.16)

which gives

\[
F(t) = \frac{r}{r - 1} \left[e^{(r-1)t/\tau_{\text{cav}} - 1}\right],
\]  (5.17)

where \( t = 0 \) now corresponds to the instant of the Q-switch.

The intensity thus increases exponentially with a time constant \( \tau_{\text{dl}} = \tau_{\text{cav}}/(r - 1) \). Most often, such lasers operate in favorable conditions in which
5.3. Q-SWITCHED LASER

$r \gg 1$ and $\tau_d \ll \tau_{cav}$. The build-up time of the pulse $t_d$, which can be defined as the time duration between the Q-switch time and the time at which the intensity is equal to the saturation intensity, can be much shorter than the lifetime of the photons in the cavity:

$$t_d = \frac{\tau_{cav}}{r - 1} \ln \left( \frac{r}{r - 1} F_{sat} \right) \approx \frac{\tau_{cav}}{r - 1} \ln F_{sat} .$$

(5.18)

### 5.3.3.3 Middle of the pulse

During the pulse, the intensity is so strong that the pumping and decay terms in equation (5.2) can be neglected:

$$\frac{d}{dt} \Delta N = -2 \kappa F \Delta N .$$

(5.19)

Even if the time evolutions $\Delta N(t)$ and $F(t)$ cannot be derived, we can obtain the function $F(\Delta N)$ by dividing (5.1) by (5.19):

$$\frac{dF}{d\Delta N} = \frac{1}{\kappa \tau_{cav} \Delta N} - 1 .$$

(5.20)

Let us integrate this equation between $t = 0$ (Q-switch time) and an arbitrary time $t$ during the pulse, leading to:

$$F(t) - F(0) = \Delta N_i - \Delta N(t) - \Delta N_{th} \ln \frac{\Delta N_i}{\Delta N(t)} ,$$

(5.21)

because the population at threshold obeys $\Delta N_{th} = 1/\tau_{cav} \kappa$. Besides, we can take $F(0) = 0$ in this equation.

Let us call $t_f$ an instant located at the end of the pulse. We then introduce the quantity:

$$\eta = \frac{\Delta N_i - \Delta N(t_f)}{\Delta N_i} ,$$

(5.22)

which is the energetic yield of the pump energy. Equation (5.21) taken at $t = t_f$ reads, since $F(t_f) \approx 0$:

$$0 = \Delta N_i - \Delta N(t_f) - \Delta N_{th} \ln \frac{\Delta N_i}{\Delta N(t_f)} ,$$

(5.23)

or, using equations (5.15) and (5.22):

$$1 - \eta = e^{-r \eta} .$$

(5.24)
Figure 5.7: Graphic solving of equation (5.24) for \( r = 4 \). The solution, which is the intersection of the two curves, is very close to 1.

Equation (5.24) cannot be solved analytically. However, a graphic solving (see figure 5.7) shows that \( \eta \) is very close to 1 as soon as \( r \) is equal to a few units. The total energy extracted out of the active medium is thus given by:

\[
U_{\text{tot}} = \eta \hbar \omega \Delta N_i = \eta \hbar \omega r \Delta N_{\text{th}} \simeq \hbar \omega r \Delta N_{\text{th}} .
\] (5.25)

Besides, the number of photons in the cavity reaches its maximum value \( F_{\text{max}} \) when \( dF/dt = 0 \), i.e., using equation (5.1), when \( \Delta N = \Delta N_{\text{th}} \). Using equation (5.21), we get:

\[
F_{\text{max}} = \Delta N_{\text{th}} (r - 1 - \ln r) .
\] (5.26)

The maximum power dissipated by the laser is thus given by:

\[
P_{\text{dis}}_{\text{max}} = \frac{\hbar \omega}{\tau_{\text{cav}}} F_{\text{max}} = \frac{\hbar \omega}{\tau_{\text{cav}}} \Delta N_{\text{th}} (r - 1 - \ln r) .
\] (5.27)

and the part of this power which is available at the laser output (supposed to be based on a ring cavity) through a coupler of transmission \( T \) is:

\[
P_{\text{out}}_{\text{max}} = \frac{T}{\Pi} P_{\text{dis}}_{\text{max}} = \frac{c_0 T}{n_0 L_{\text{cav}}} \hbar \omega \Delta N_{\text{th}} (r - 1 - \ln r) .
\] (5.28)

Finally, the duration \( \Delta t \) of the pulse is roughly given by the ratio between the dissipated energy and the maximum dissipated power:

\[
\Delta t \simeq \frac{U_{\text{tot}}}{P_{\text{dis}}_{\text{max}}} = \frac{\eta r}{r - 1 - \ln r} \tau_{\text{cav}} ;
\] (5.29)

which is, as expected, of the order of magnitude of the photon lifetime in the cavity.

\(^1\)This change of variable is possible because \( \Delta N(t) \) evolve in a monotonous manner during the pulse, as can be seen in figure 5.5.
5.3. Q-SWITCHED LASER

5.3.3.4 End of the pulse

Since the energy yield $\eta$ is very close to 1 (see figure 5.7), the pulse efficiently empties the energy which has been stored in the population inversion during the pumping phase. In other words, at the end of the pulse, we have $\Delta N = \Delta N(t_f) \approx 0$. We are thus left with only the photon lifetime term in equation (5.1):

$$\frac{dF}{dt} = -\frac{F}{\tau_{cav}}.$$  \hspace{1cm} (5.30)

The decay of the pulse intensity is thus exponential with a decay rate $\tau_{cav}$. If we compare this evolution with the laser build-up, we can see that the temporal pulse shape is asymmetric. This property is indeed experimentally verified.

5.3.4 Example.

Let us consider for example a Nd:YAG laser ($\lambda = 1.064 \, \mu m$) with a cavity optical length equal to $L_{cav, opt} = 0.5 \, m$. We suppose that the cavity losses per round-trip are equal to 10% ($\Pi = 0.1$) among which 5% are due to the output coupler transmission ($T = 0.05$). The lifetime of the photons in the cavity is $\tau_{cav} = L_{cav, opt}/c_0 \Pi = 17 \, ns$. The cross section of the considered transition is roughly given by $\sigma \simeq 3 \times 10^{-19} \, cm^2$ and we suppose that the beam has a section area $S = 3 \, mm^2$. If we pump this laser 10 times above threshold ($r = 10$), we thus expect:

$$\kappa = \frac{\sigma}{S} \frac{c_0}{L_{cav, opt}} = 6 \times 10^{-9} \, s^{-1},$$  \hspace{1cm} (5.31)

$$\Delta N_{th} = \frac{1}{\kappa \tau_{cav}} = 10^{16} \, \text{atoms},$$  \hspace{1cm} (5.32)

$$U_{tot} \simeq \frac{h\omega r \Delta N_{th}}{\kappa} = 19 \, mJ,$$  \hspace{1cm} (5.33)

$$F_{max} = \Delta N_{th} (r - 1 - \ln r) \simeq 7 \times 10^{16} \, \text{photons},$$  \hspace{1cm} (5.34)

$$P_{out}^{max} = \frac{h\omega}{\Pi \tau_{cav}} F_{max} = 380 \, kW,$$  \hspace{1cm} (5.35)

$$\Delta t \simeq \frac{U_{tot}}{\Pi P_{out}^{max}} = 25 \, ns.$$  \hspace{1cm} (5.36)

Focused to a $300 \, \mu m^2$ spot, such a pulse leads to a peak intensity of the order of $10^{15} \, W/m^2$. 
5. TRANSIENT AND Q-SWITCHED OPERATIONS
Chapter 6

Frequency and intensity noises

We have seen in the preceding chapters that spontaneous emission plays an important role in the laser start-up. Indeed, it provides the “first photon” which can then be amplified by stimulated emission in the active medium and allows the laser to start oscillating. However, once the laser has reached its steady-state regime, spontaneous emission still occurs since some atoms are still in the upper level of the transition. This spontaneous emission is, to several respects, a random process: it is emitted at random times with a random direction, a random phase, and a random polarization. It is consequently not always emitted in the laser mode. However, when it falls into the laser mode, it adds to the intracavity field with a random phase. It thus leads to a stochastic evolution of the laser field and thus to a noise.

It is not possible to rigorously describe spontaneous emission in the framework of the semi-classical model that we use in this course. A completely quantum approach lies beyond the scope of this course. However, we can introduce spontaneous emission as a random force in the equation of evolution of the intracavity classical field. This will allow us to obtain a “Langevin equation” to describe the evolution of the laser field and to derive the phase and amplitude noises of a class-A laser. We will also extend these results to the case of a class-B laser.

6.1 Langevin equation for the class-A laser

6.1.1 Equation of evolution of the intensity

Let us consider a single-frequency ring laser containing a homogeneously broadened active medium. Then we have seen that the equations of evolution of the intensity $I$ and of the population inversion $\Delta N$ are the Statz and de
Let us first suppose that our laser is a class-A laser, i.e., that the lifetime \( \tau \) of the population inversion is much shorter than the lifetime \( \tau_{\text{cav}} \) of the photons in the cavity. This typically corresponds to the case of gas lasers or dye lasers. In this case, the population inversion “instantaneously” reacts to the variations of the intensity: equation (6.2) can thus be adiabatically eliminated, leading to:

\[
\Delta N(t) = \frac{\Delta N_0}{1 + \frac{I(t)}{I_{\text{sat}}}}.
\]

By injecting this equation into equation (6.1), we obtain the equation of evolution of the intensity of a monomode class-A laser with a homogeneously broadened gain medium:

\[
\frac{dI}{dt} = \frac{I}{\tau_{\text{cav}} \left( \frac{r}{1 + \frac{I(t)}{I_{\text{sat}}}} - 1 \right)},
\]

where we have introduced the excitation ratio \( r = \Delta N_0/\Delta N_{\text{th}} \) of the laser. The laser is said to be “\( r \) times above threshold.” The non-zero steady-state solution of this equation is given by:

\[
I_{\text{ON}} = I_{\text{sat}} (r - 1).
\]

### 6.1.2 Equation of evolution for the field

It is worth noticing that equation (6.4) contains only the intensity of the field: the phase plays no role. If we want to introduce the laser phase, we may write the intracavity field in the following manner:

\[
E(r, t) = u(r) A(t) e^{-i\omega t} + \text{c.c.},
\]

where \( u(r) \) contains the polarization and the intracavity spatial distribution of the field in the oscillating mode (for example \( u(r) = \varepsilon_x^* e^{ikz} \) for a plane wave polarized along \( x \) and propagating along \( z \)), and where \( A(t) \) is the slowly varying complex amplitude of the laser field. The intensity is given...
by \( I(t) = 2c_0\varepsilon_0|A(t)|^2 \) (we take \( n_0 = 1 \) in this chapter) and equation (6.4) is replaced by:

\[
\frac{dA}{dt} = \frac{A}{2\tau_{cav}} \left( \frac{r}{1 + \frac{2c_0\varepsilon_0|A|^2}{I_{sat}}} - 1 \right). \tag{6.7}
\]

By writing the complex amplitude \( A(t) \) as:

\[
A(t) = |A(t)|e^{i\varphi(t)}, \tag{6.8}
\]

we obtain the equations of evolution for the modulus and the phase of the amplitude of the laser field:

\[
\frac{d}{dt}|A(t)| = \frac{|A(t)|}{2\tau_{cav}} \left( \frac{r}{1 + \frac{2c_0\varepsilon_0|A|^2}{I_{sat}}} - 1 \right), \tag{6.9}
\]

\[
\frac{d\varphi}{dt} = 0. \tag{6.10}
\]

Equation (6.9), which is equivalent to equation (6.4), shows that the steady-state solution corresponds to

\[
A_{ON} = \sqrt{r - 1} \sqrt{\frac{I_{sat}}{2c_0\varepsilon_0}}, \tag{6.11}
\]

i. e., to any points on the circle of figure 6.1. Equation (6.10) shows that nothing imposes the phase of the laser field; any point on the circle of figure 6.1 is a valid steady-state solution.

Figure 6.1: Representation of the complex amplitude \( A(t) \) of the laser field. The steady-state solutions correspond to the circle in the figure.

Comment:
Equation (6.7) could have been directly obtained by adiabatically eliminating $\mathcal{P}$ and $\Delta n$ in equations (2.23-2.25).

### 6.1.3 Heuristic introduction of spontaneous emission

Spontaneous emission is introduced as an extra term $\zeta_A(t)$ in equation (6.7):

$$
\frac{dA}{dt} = \frac{A}{2\tau_{\text{cav}}} \left( \frac{r}{1 + \frac{2\omega_{\text{res}}|A|^2}{I_{\text{sat}}} - 1} \right) + \zeta_A(t) .
$$

(6.12)

$\zeta_A(t)$ is the random complex amplitude corresponding to the part of spontaneous emission which falls into the laser mode. Equation (6.12) is called the Langevin equation of evolution of the complex amplitude of the laser field. Since spontaneous emission is emitted with a random phase, this amplitude has a zero average at all times:

$$
\langle \zeta_A(t) \rangle = 0 ,
$$

(6.13)

where $\langle \ldots \rangle$ holds for the statistical average. The spontaneous emission power is given by its auto-correlation:

$$
\Gamma_{\zeta_A}(T) = \langle \zeta_A^*(t)\zeta_A(t + T) \rangle = N_2^2 \Gamma_{\text{sp}} (A_{1\text{phot}})^2 \delta(T) .
$$

(6.14)

This equation simply means that the power of spontaneous emission falling in the laser mode is proportional to the total number $N_2$ of atoms in the upper level of the laser transition which are able to interact with the laser mode (and are thus able to spontaneously emit into this mode), to $\Gamma_{\text{sp}}$ the rate of spontaneous emission into the laser mode for an atom in level 2, and of course to $(A_{1\text{phot}})^2$ which is the square of the field amplitude corresponding to one photon in the laser mode. Equation (6.14) can be qualitatively justified by writing that the extra amplitude added to the field by spontaneous emission reads:

$$
A_{\text{sp}}(t) = \sum_j A_{1\text{phot}} H(t - t_j)e^{i\varphi_j} ,
$$

(6.15)

where $H$ is the Heaviside function, where the times $t_j$ at which spontaneous emission occurs are randomly distributed with a mean rate of $N_2 \Gamma_{\text{sp}}$ emissions per second and where the phases $\varphi_j$ are independent random variables equally distributed in $[0, 2\pi]$. The autocorrelation of the Langevin force can then be deduced from equation (6.15):

$$
\Gamma_{\zeta_A}(T) = A_{1\text{phot}}^2 \left\langle \sum_j \sum_k e^{i(\varphi_k - \varphi_j)} \delta(t - t_j) \delta(t - t_k + T) \right\rangle ,
$$

(6.16)
because the derivative of $H$ is the Dirac function $\delta$. Since $\varphi_j$ and $\varphi_k$ are independent variables, we have $\langle e^{i(\varphi_k-\varphi_j)} \rangle = \delta_{jk}$. We thus obtain:

$$\Gamma_{\zeta_A}(T) = A_{1\text{phot}}^2 \sum_j \langle \delta(t - t_j)\delta(t - t_j + T) \rangle .$$  \hfill (6.17)

Finally, by replacing the sum $\sum_j$ in equation (6.17) by the integral $N_2 \Gamma_{sp} \int dt_j$ and by integrating we eventually obtain equation (6.14). It is worth noticing that the fact that the autocorrelation of equation (6.14) is proportional to $\delta(T)$ comes from the fact that we suppose that the successive spontaneous emissions, and in particular their phases, are independent. Finally, the fact that the statistical properties of $\zeta_A(t)$ do not depend on $t$ shows that this is a stationary process.

In order to proceed, we need to know the spontaneous emission rate $\Gamma_{sp}$ for an atom in level 2. To this aim, we admit the following result, which is given by the theory of spontaneous emission provided by quantum optics (i.e., in the presence of quantized electromagnetic fields): \textit{the rate of spontaneous emission is equal to the rate of stimulated emission in the presence of one photon in the considered mode.} Using the last term in equation (6.2) for one atom with the intensity corresponding to one photon in the mode and noted $I_{1\text{phot}}$, we can see that this spontaneous emission rate is then, in the case of a four level system, given by:

$$\Gamma_{sp} = \frac{1}{\tau} \frac{I_{1\text{phot}}}{I_{\text{sat}}} = \frac{1}{\tau} \frac{c_{0}\hbar}{V_{\text{cav}}} \frac{\hbar\omega}{\sigma} = \frac{c_{0}\sigma}{V_{\text{cav}}} ,$$  \hfill (6.18)

where $V_{\text{cav}}$ is the volume occupied by the laser mode in the cavity and where we have used equation (2.31) to link the intensity with the number of intra-cavity photons. Then, in equation (6.14), we have, for a four-level system:

$$N_2 \Gamma_{sp} = \Delta N \Gamma_{sp} = V_a \Delta n \Gamma_{sp} ,$$  \hfill (6.19)

where $V_a$ is the volume occupied by the laser mode in the active medium. In the vicinity of the laser steady-state solution we have $\Delta n \simeq \Delta n_{\text{th}} = \Pi / (L_a \sigma)$ which can be inserted into equation (6.19) to lead:

$$N_2 \Gamma_{sp} = V_a \frac{\Pi c_{0}\sigma}{L_a \sigma V_{\text{cav}}} = \frac{\Pi c_{0}}{L_{\text{cav}}} = \frac{1}{\tau_{\text{cav}}} .$$  \hfill (6.20)

More generally, in the case of an active medium which cannot be treated as a four-level system, we have:

$$N_2 \Gamma_{sp} = \frac{N_2}{\Delta N} \frac{1}{\tau_{\text{cav}}} .$$  \hfill (6.21)

In the following, we will focus on the case of a four-level system and we will hence use equation (6.20).
6.2 Amplitude and phase noises of a class-A laser

The Langevin equation (6.12) can be transformed into a Langevin equation for the field amplitude and a Langevin equation for the phase of the field. For example, if we consider the small fluctuations of the field around its steady-state solution, we can write the field amplitude as:

\[ \mathcal{A}(t) = [\mathcal{A}_{ON} + a(t)] e^{i\varphi(t)}, \]  

(6.22)

where \( a(t) \) is real and small compared with \( \mathcal{A}_{ON} \). We thus replace a complex stochastic process \( \mathcal{A}(t) \) by two real stochastic processes \( a(t) \) and \( \varphi(t) \). Keeping only first order terms, equation (6.12) becomes:

\[ \frac{da}{dt} + i \mathcal{A}_{ON} \frac{d\varphi}{dt} + \frac{r-1}{r} a \tau_{cav} = \zeta_A(t) e^{-i\varphi(t)}. \]

(6.23)

The real and imaginary parts of equation (6.23) lead to the Langevin equations governing the amplitude and the phase of the laser:

\[ \frac{da}{dt} + \frac{r-1}{r} a \tau_{cav} = \zeta_1(t), \]

(6.24)

\[ \frac{d\varphi}{dt} = \frac{1}{\mathcal{A}_{ON}} \zeta_2(t). \]

(6.25)

Figure 6.2: The Langevin force \( \zeta_A(t) \) is decomposed into a radial component \( \zeta_1(t) \) which acts on the laser amplitude and a tangential component \( \zeta_2(t) \) which makes the phase diffuse.
6.2. NOISES OF A CLASS-A LASER

\( \zeta_1(t) \) and \( \zeta_2(t) \) are the real and imaginary parts of \( \zeta_A(t)e^{-i\varphi(t)} \). They correspond to the components of the spontaneous emission which are respectively parallel and perpendicular to the laser field, as shown in figure 6.2. Since there is no preferred direction in the complex plane, they have the same statistical properties as the real and imaginary parts of \( \zeta_A(t) \). Thus, using equations (6.14) and (6.20), we get:

\[
\langle \zeta_1(t) \rangle = \langle \zeta_2(t) \rangle = 0, \quad (6.26)
\]

\[
\Gamma_{\zeta_1}(T) = \Gamma_{\zeta_2}(T) = \frac{1}{2\tau_{cav}}(A_{1\text{phot}})^2\delta(T) = 2D_a\delta(T), \quad (6.27)
\]

where \( D_a = \frac{1}{4\tau_{cav}}(A_{1\text{phot}})^2 \) is the diffusion coefficient for the amplitude.

### 6.2.1 Amplitude noise

The power spectral density of the laser amplitude noise can be obtained by deriving the autocorrelation of \( a(t) \) and by taking its Fourier transform. Equation (6.24) can be formally integrated to give:

\[
a(t) = \int_{-\infty}^{t} dt' \zeta_1(t')e^{-\beta(t-t')}, \quad (6.28)
\]

where we have introduced the damping coefficient \( \beta = (r - 1)/r\tau_{cav} \). The autocorrelation function of \( a(t) \) is then equal to:

\[
\Gamma_a(T) = \langle a(t)a(t+T) \rangle = \int_{-\infty}^{t} dt' \int_{-\infty}^{t+T} dt'' 2D_a\delta(t' - t'')e^{-\beta(2t+T-t'-t'')}
\]

\[
= \frac{D_a}{\beta} e^{-\beta|T|}. \quad (6.29)
\]

Using the Wiener-Khintchine theorem, we get the spectrum of the laser amplitude fluctuations:

\[
S_a(\Omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\Omega T} \Gamma_a(T) dT = \frac{D_a}{\beta} \frac{1}{\pi} \frac{\beta}{\beta^2 + \Omega^2}. \quad (6.30)
\]

The variance of these fluctuations is given by the value of the autocorrelation at \( T = 0 \):

\[
\sigma_a^2 = \Gamma_a(0) = \frac{D_a}{\beta}. \quad (6.31)
\]
By introducing the intracavity photon number \( F_{\text{ON}} = \frac{A_{\text{ON}}^2}{A_{\text{phot}}^2} \), we derive the relative fluctuations of the intensity:

\[
\frac{\sigma_a^2}{A_{\text{ON}}^2} = \frac{D_a}{\beta} \frac{1}{A_{\text{ON}}^2} = \frac{1}{4\tau_{\text{cav}}} \frac{A_{\text{phot}}^2 r\tau_{\text{cav}}}{A_{\text{ON}}^2} \frac{r}{r-1} = \frac{r}{4(r-1) F_{\text{ON}}}.
\] (6.32)

Consequently, when the laser is far above threshold \( (r \gg 1) \), the relative amplitude fluctuations are \( \sigma_a/A_{\text{ON}} \approx 1/(2\sqrt{F_{\text{ON}}}) \). The relative intensity noise (RIN) spectrum, given by \( S_a(\Omega) / A_{\text{ON}}^2 \) with equation (6.30), is Lorentzian with a width \( 2\beta \), close to the inverse of the photon lifetime in the cavity: the influence of spontaneous emission on the intensity is filtered by the response time of the cavity. This is consistent with the transient behavior of class-A lasers that we have studied in section 5.1 (see in particular equation 5.8).

**Comments:**

1. The reader should be careful not to confuse the *amplitude fluctuations* derived here with the *shot noise* that appears when a laser field of amplitude \( A_{\text{ON}} \) is detected. This shot noise is indeed due to the discrete generation of the carriers in the detector, contrary to the amplitude noise calculated here which has nothing to do with the quantum nature of light and simply comes from the stochastic nature of the spontaneous emission that falls in the laser mode.

2. The amplitude noise derived here takes into account only the fluctuations created by spontaneous emission. Any other source of fluctuations, such as, e.g., fluctuations in the pumping rate \( \Delta N_0 \) (pump laser noise) or in the losses (fluctuation of the term \(-1/\tau_{\text{cav}} \) due due for example to mechanical or acoustic fluctuations of the cavity) can be taken into account using the same Langevin equation formalism developed here. The spectrum of the corresponding Langevin force will then reproduce the spectrum of the considered fluctuations.

### 6.2.2 Phase noise

Contrary to the laser amplitude which is constantly attracted towards its steady-state value \( A_{\text{ON}} \) (see the return term in equation (6.24), thus attracting the laser field towards the circle of figure 6.2), the phase can evolve freely along the circle of figure 6.2 under the influence of the Langevin force \( \zeta/\dot{A}_{\text{ON}} \) of equation (6.25). Actually, this equation describes a Brownian motion for the phase \( \varphi(t) \) under the influence of this Langevin force. The phase diffusion
coefficient associated with this random walk is given by:

\[ D_\phi = \frac{D_a}{A_{ON}^2} = \frac{1}{4F_{cav}}. \] (6.33)

\( \varphi(t) \) will thus diffuse along the whole circle according to a Brownian motion and will thus not obey a stationary process. To deal with the technical difficulty related with the non zero initial value of the phase, we introduce the phase excursion \( \delta \varphi(t) \):

\[ \delta \varphi(t) = \varphi(t) - \varphi(0), \] (6.34)

which verifies \( \delta \varphi(0) = 0 \). \( \varphi(0) \) is the value of the phase randomly chosen by the laser on the circle when it is switched on. We can formally integrate equation (6.25) between 0 and \( t \), leading to:

\[ \delta \varphi(t) = \int_0^t dt' \frac{\xi_2(t')}{A_{ON}}, \] (6.35)

which leads, using equation (6.27), to the variance of the phase excursion:

\[ \langle \delta \varphi^2(t) \rangle = \int_0^t dt' \int_0^t dt'' \frac{\langle \xi_2(t') \xi_2(t'') \rangle}{A_{ON}^2} = \int_0^t dt' 2D_\varphi = 2D_\varphi t. \] (6.36)

One can thus see that the laser phase obeys a 1D random walk with diffusion coefficient \( D_\varphi \). The laser frequency noise is given by:

\[ \delta \nu(t) = \frac{1}{2\pi} \delta \varphi(t) = \frac{\xi_2(t)}{2\pi A_{ON}}. \] (6.37)

Consequently, the laser frequency noise spectrum is given by:

\[ S_{\delta \nu}(\Omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dT e^{i\Omega T} \langle \delta \nu(t) \delta \nu(t + T) \rangle = \frac{2D_a}{8\pi^3 A_{ON}^2} = \frac{1}{16\pi^3 \tau_{cav} F_{ON}}. \] (6.38)

We can thus notice that spontaneous emission induces a white frequency noise.

### 6.2.3 Laser linewidth

Because of its amplitude and phase fluctuations, the laser field itself becomes a stochastic process. The laser field will consequently not be purely monochromatic but will have a finite linewidth. To derive the laser field spectrum, one
can usually neglect the amplitude fluctuations and consider only the influence of phase diffusion.

Let us remind that the positive frequency part of the laser field reads:

$$E^{(+)}(t) = A_{ON} e^{-i\omega t} e^{i\varphi(t)}.$$  \hspace{1cm} (6.39)

Once again, the spectrum of this laser field is obtained using the Wiener-Khintchine theorem:

$$S_{E^{(+)}(\omega')} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dT \langle E^{(+)}(t + T)[E^{(+)}(t)]^* \rangle e^{i\omega'T}$$

$$= \frac{A_{ON}^2}{2\pi} \int_{-\infty}^{\infty} dT e^{i(\omega' - \omega)T} \langle e^{i[\varphi(t + T) - \varphi(t)]} \rangle$$

$$= \frac{A_{ON}^2}{2\pi} \int_{-\infty}^{\infty} dT e^{i(\omega' - \omega)T} e^{-\frac{1}{2} \langle \delta\varphi^2(T) \rangle}$$

$$= \frac{A_{ON}^2}{2\pi} \int_{-\infty}^{\infty} dT e^{i(\omega' - \omega)T} e^{-D\varphi|T|}$$

$$= \frac{A_{ON}^2}{2\pi} \frac{2D\varphi}{D^2\varphi + (\omega' - \omega)^2},$$  \hspace{1cm} (6.40)

where we have used the fact that if $\delta\varphi(T)$ is a centered Gaussian process, then $\langle e^{i\delta\varphi(T)} \rangle = e^{-\frac{1}{2} \langle \delta\varphi^2(T) \rangle}$.

This spectrum is Lorentzian, as shown in figure 6.3. Its FWHM, called the Schawlow-Townes linewidth, is given by:

$$\Delta\nu = \frac{D\varphi}{\pi} = \frac{1}{4\pi F\tau_{cav}} = \frac{h\nu}{4\pi \tau_{cav}^2 P_{out}},$$  \hspace{1cm} (6.41)

where $P_{out}$ is the laser output power provided the only cavity losses are due to the transmission of the output coupler. The interpretation of equation (6.41) is interesting: we can see that the laser linewidth is given, to a numerical factor, by the cold cavity linewidth $1/(2\pi\tau_{cav})$ divided by the number $F$ of photons inside this cavity.

Depending on the laser type, the Schawlow-Townes linewidth can vary by several orders of magnitude. For example, for a He-Ne laser ($\lambda = 633$ nm) with a 20-cm cavity length, with 1% losses per round-trip and emitting a 1 mW power, we obtain a Schawlow-Townes linewidth equal to 70 mHz. A semiconductor laser emitting the same power at the same wavelength but
6.3. CASE OF A CLASS-B LASER

Figure 6.3: Spectrum of the laser field. The full width at half maximum is the so-called Schawlow-Townes linewidth.

with a 1-mm long cavity having 50% losses per round-trip will exhibit a Schawlow-Townes linewidth of 7 MHz.

Comments:

1. Here again, the Schawlow-Townes linewidth must be seen as a fundamental limit for the laser linewidth. There exist many other mechanisms that can broaden the laser line, such as, e. g., the fluctuations of the cavity length that induce fluctuations of the laser frequency known as the laser “frequency jitter”.

2. The equation (6.41) giving the Schawlow-Townes linewidth must often be completed by many other factors which are not discussed here.

6.3 General case and application to the class-B laser

Up to now, we have limited our discussion to the case of class-A lasers, i. e. to the case where $\tau_{\text{cav}} \gg \tau$. We shall see in this section that the above developments can be generalized, leading in particular to the intensity noise spectrum for a class-B laser.
6.3.1 Derivation of the Langevin equations

In this case, equation (6.12) must be replaced by the following equations:

\[
\frac{dA}{dt} = \frac{A}{2\tau_{\text{cav}}} \left( \frac{\Delta N}{\Delta N_{\text{th}}} - 1 \right) + \zeta_A(t), \quad (6.42)
\]

\[
\frac{d\Delta N}{dt} = \frac{1}{\tau} \left( \Delta N_0 - \Delta N - \frac{F}{F_{\text{sat}}} \Delta N \right) + \zeta_{\Delta N}(t). \quad (6.43)
\]

We will discuss below the origin of the new Langevin force \( \zeta_{\Delta N}(t) \). Let us focus on the intensity noise only. In this case, it is more convenient to write equations (6.42) and (6.43) for the number \( F \) of photons in the cavity and the number \( \Delta N \) of atoms in population inversion in the laser mode volume.

Since \( F = \frac{IV_{\text{cav}}}{c_0\hbar\omega} \) with \( I = 2c_0\varepsilon_0|A|^2 \), we obtain the following equation of evolution of the number of photons:

\[
\frac{dF}{dt} = \frac{F}{\tau_{\text{cav}}} \left( \frac{\Delta N}{\Delta N_{\text{th}}} - 1 \right) + \zeta_F(t), \quad (6.44)
\]

with

\[
\zeta_F(t) = \frac{2\varepsilon_0V_{\text{cav}}}{\hbar\omega} \left( A_{\text{ON}} e^{-i\phi} \zeta_A(t) + \text{c.c.} \right). \quad (6.45)
\]

One thus obtains the autocorrelation of \( \zeta_F(t) \):

\[
\langle \zeta_F(t) \zeta_F(t + T) \rangle = \frac{2F_{\text{ON}}}{\tau_{\text{cav}}} \frac{1}{\tau} \delta(T), \quad (6.46)
\]

where \( F_{\text{ON}} = \frac{2\varepsilon_0V_{\text{cav}}}{\hbar\omega} A_{\text{ON}}^2 \) is the average number of photons in the cavity in steady-state regime. Finally, we can re-write the equations of evolution of the laser (6.42, 6.43) as functions of the number of photons in the cavity and the number of atoms in population inversion in the laser mode according to:

\[
\frac{dF}{dt} = \frac{F}{\tau_{\text{cav}}} \left( \frac{\Delta N}{\Delta N_{\text{th}}} - 1 \right) + \zeta_F(t), \quad (6.47)
\]

\[
\frac{d\Delta N}{dt} = \frac{1}{\tau} \left( \Delta N_0 - \Delta N - \frac{F}{F_{\text{sat}}} \Delta N \right) + \zeta_{\Delta N}(t), \quad (6.48)
\]

where \( F_{\text{sat}} = \frac{V_{\text{cav}}}{c_0\hbar\omega} I_{\text{sat}} \). The autocorrelation of the Langevin force \( \zeta_F(t) \) is given by equation (6.46). Its power spectral density is thus given by:

\[
S_{\zeta_F}(\Omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dT e^{i\Omega T} 2F_{\text{ON}} \frac{1}{\tau_{\text{cav}}} \delta(T) = \frac{F_{\text{ON}}}{\pi \tau_{\text{cav}}}. \quad (6.49)
\]
Let us now consider the Langevin force $\zeta_{\Delta N}(t)$ which acts on the evolution of the population inversion in equation (6.48). We can obtain the expression of its spectrum by analogy with equation (6.49). First, let us remember that each spontaneously emitted photon falling into the laser mode corresponds to a decrease of $\Delta N$ by one unit. Consequently, $\Delta N$ experiences the same spontaneous emission noise as $F$ (with a minus sign). Thus the term $\frac{F_{\text{ON}}}{\pi \tau_{\text{cav}}}$ must also be present in the spectrum of $\zeta_{\Delta N}(t)$. Besides, we notice that the term $\frac{F_{\text{ON}}}{\pi \tau_{\text{cav}}}$ describes the dissipation induced by the cavity losses. Since the population inversion undergoes a similar dissipation mechanism given by the term $-\Delta N/\tau$ in equation (6.48), the random nature of this population relaxation must also lead to the existence of the same type of term in the Langevin force $\zeta_{\Delta N}(t)$. To summarize, we have:

$$S_{\zeta_{\Delta N}}(\Omega) = \frac{1}{\pi} \left( \frac{F_{\text{ON}}}{\tau_{\text{cav}}} + \frac{\Delta N_{\text{th}}}{\tau} \right).$$ \hspace{1cm} (6.50)

Moreover, the correlation spectrum between the two Langevin forces is given by:

$$S_{\zeta_F \zeta_{\Delta N}}(\Omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dT e^{i\Omega T} \langle \zeta_{\Delta N}(t) \zeta_F(t + T) \rangle = -\frac{F_{\text{ON}}}{\pi \tau_{\text{cav}}},$$ \hspace{1cm} (6.51)

since the effects of spontaneous emission are exactly opposite for the number of photons and the population inversion.

**6.3.2 Application to the calculation of the RIN of class-A and class-B lasers**

Let us now focus on the fluctuations of the state of the laser around its steady-state solution by writing:

$$F(t) = F_{\text{ON}} + \delta F(t),$$ \hspace{1cm} (6.52)

$$\Delta N(t) = \Delta N_{\text{th}} + \delta N(t),$$ \hspace{1cm} (6.53)

where $\delta F(t)$ and $\delta N(t)$ are supposed to be small with respect to $F_{\text{ON}}$ and $\Delta N_{\text{th}}$. By injecting (6.52) and (6.53) into equation (6.47) and keeping only the lowest order terms, we have:

$$\delta F = \frac{F_{\text{ON}}}{\tau_{\text{cav}}} \frac{\delta N}{\Delta N_{\text{th}}} + \zeta_F(t).$$ \hspace{1cm} (6.54)
We then use equations (2.31), (2.36), (2.56), (3.5), and (3.8) to write:

\[ F_{ON} = \frac{V_{cav}}{c_0 \hbar \omega} I_{ON} = \frac{V_{cav}}{c_0 \hbar \omega} I_{sat} (r - 1) \]
\[ = \frac{V_{cav} \hbar \omega}{c_0 \hbar \omega} \sigma \tau (r - 1) = \frac{V_{cav}}{c_0 \sigma \tau} (r - 1) \]  
(6.55)

\[ \Delta N_{th} = \frac{\kappa}{\kappa_{cav}} \]
(6.56)

\[ \kappa = \frac{\sigma c_0 V_{cav}}{V_{cav}} \]  
(6.57)

Consequently, equation (6.54) can be simplified to read:

\[ \delta F = \frac{r - 1}{\tau} \delta N + \zeta_F(t) \]  
(6.58)

Similarly, by injecting (6.52) and (6.53) into equation (6.48), keeping only lowest order terms and using the fact that \( \Delta N_{th} = \frac{\zeta_F}{F_{sat}} \), we get:

\[ \delta N = -\frac{r}{\tau} \delta N - \frac{1}{\tau_{cav}} \delta F + \zeta_{\Delta N}(t) \]  
(6.59)

In order to solve equations (6.58) and (6.59), we take their Fourier transforms:

\[ i \Omega \tilde{\delta F}(\Omega) + \frac{r - 1}{\tau} \tilde{\delta N}(\Omega) = -\tilde{\zeta}_F(\Omega) \]  
(6.60)

\[ \frac{1}{\tau_{cav}} \tilde{\delta F}(\Omega) + \left( \frac{r}{\tau} - i \Omega \right) \tilde{\delta N}(\Omega) = \tilde{\zeta}_{\Delta N}(\Omega) \]  
(6.61)

By eliminating \( \tilde{\delta N}(\Omega) \), one readily obtains:

\[ \left[ \Omega^2 - \frac{r - 1}{\tau_{cav}} + i \frac{\Omega}{\tau} \right] \tilde{\delta F}(\Omega) = -\frac{r - 1}{\tau} \tilde{\zeta}_{\Delta N}(\Omega) - \left( \frac{r}{\tau} - i \Omega \right) \tilde{\zeta}_F(\Omega) \]  
(6.62)

To obtain the spectrum, we take the square modulus of equation (6.62) using (6.49-6.51), leading to:

\[ S_{\delta F}(\Omega) = \frac{F_{ON}}{\pi \tau_{cav}} \frac{\frac{r}{\tau} + \Omega^2}{\left( \Omega^2 - \frac{r - 1}{\tau_{cav}} \right)^2 + \frac{r^2 \Omega^2}{\tau^2}} \]  
(6.63)

Then, using the fact that the output coupling losses are the only cavity losses, the output power is given by \( P_{out} = \frac{F_{ON} \hbar \omega}{\tau_{cav}} \), and the RIN spectrum becomes:

\[ \text{RIN}(\Omega) = \frac{S_{\delta F}(\Omega)}{F_{ON}^2} = \frac{\hbar \omega}{\pi \tau_{cav}^2 P_{out}} \frac{\frac{r}{\tau} + \Omega^2}{\left( \Omega^2 - \frac{r - 1}{\tau_{cav}} \right)^2 + \frac{r^2 \Omega^2}{\tau^2}} \]  
(6.64)
The relaxation oscillation angular frequency $\Omega_{\text{relax}} = \sqrt{\frac{r-1}{\tau \tau_{\text{cav}}}}$ can be recognized at the denominator of this expression. In the case of a class-B laser, the intensity noise thus has a resonant behavior close to the relaxation oscillation frequency. On the contrary, the behavior of the class-A laser can be recovered by supposing that $\tau_{\text{cav}} \gg \tau$ in equation (6.64). This equation then leads to:

$$\text{RIN}(\Omega) \simeq \frac{S_{F\xi}(\Omega)}{F_{\text{ON}}^2} = \frac{\hbar \omega}{\pi \tau_{\text{cav}}^2 P_{\text{out}}} \frac{1}{\Omega^2 + \left(\frac{r-1}{\tau \tau_{\text{cav}}}\right)^2} \left(\frac{\Omega^2}{\Omega^2 + \frac{r}{\tau^2}}\right)}.$$

The first term depending on $\Omega$ in equation (6.65) corresponds to a first-order filter with a cut-off angular frequency equal to $\sqrt{\frac{r-1}{\tau \tau_{\text{cav}}}}$. The second ratio varies very little in the case of a class-A laser ($\tau \ll \tau_{\text{cav}}$) and is equal to $1/\tau$ for small values of $\Omega$. We thus obtain again the fact that a class-A laser is a low-pass filter with a cut-off angular frequency given by $\sqrt{\frac{r-1}{\tau \tau_{\text{cav}}}}$.

Figure 6.4 reproduces the RIN spectra for one class-A laser and one class-B laser which differ only by the lifetimes of their population inversions. One clearly sees that the class-B laser has an intensity noise resonant around the relaxation oscillation frequency which is equal to 126 kHz in the present example, while the intensity noise of the class-A laser looks as if it was flat.
when it is represented on the same figure. Actually, as shown by equations (6.64) and (6.65), the class-B laser behaves, as long as the intensity noise is concerned, like a second order filter while the class-A laser behaves at low frequencies like a first-order filter.

Further readings:

Chapter 7

Two-frequency lasers

Up to now, we have considered almost exclusively single-frequency lasers. We have seen in particular that in the ideal case of a laser based on a homogeneously broadened active medium and a unidirectional ring cavity, gain competition leads to single-frequency operation. This is of course a very extreme situation and we have already mentioned many cases in which this approximation is not valid. Here are a few examples of situations in which mode competition can lead to a different result:

- In the case of a linear cavity, the laser modes are no longer traveling waves, but become standing waves, as shown in figure 3.10. The presence of the spatial holes burnt by these standing waves in the population inversion leads to a decrease of the competition between the modes.

- In the case of an inhomogeneously broadened active medium, depending on their frequency difference, two modes interact more or less with the same atoms. The strength of their competition will thus depend on their frequency difference.

- Let us imagine a laser cavity that can sustain the oscillation of two orthogonally polarized modes, for example two orthogonal linear polarizations. Let us also imagine that the active medium is a crystal doped with active ions that behave as linear dipoles and can exhibit different orientations. Then, depending on the orientation of the polarizations of the laser modes with respect to the active ions dipoles, the competition between the two modes can be more or less severe.

It is of course beyond the scope of the present lecture to address all these different situations. However, our aim in this chapter is to introduce a simple
model (section 7.1) for mode competition and to analyze its consequences on the behavior of two-mode lasers, either of class A (section 7.2) or B (section 7.3). In a final section (section 7.4), we also describe the very useful phenomenon of injection locking that occurs when a master laser is injected into a slave laser.

### 7.1 Self- and cross-saturation terms

In general, the problem of gain saturation in two-mode lasers is very complicated. Our aim here is to heuristically introduce the minimum of formalism leading to the correct description of the physical phenomena observed in actual systems. Let us consider a laser sustaining the oscillation of two modes labeled 1 and 2. These two modes can be two different longitudinal modes, two different transverse modes, two counterpropagating modes in a ring laser, or two different polarization modes. If we note $F_1$ and $F_2$ the numbers of photons of these two modes, and if we suppose that these two modes take their gains from two independent population inversion reservoirs $\Delta N_1$ and $\Delta N_2$, then the rate equations for this laser read:

\[
\frac{\mathrm{d}F_1}{\mathrm{d}t} = -\frac{F_1}{\tau_{\text{cav}1}} + \kappa_1 F_1 \Delta N_1 ,
\]

\[
\frac{\mathrm{d}\Delta N_1}{\mathrm{d}t} = \frac{1}{\tau} (\Delta N_{01} - \Delta N_1) - 2^* \kappa_1 F_1 \Delta N_1 ,
\]

\[
\frac{\mathrm{d}F_2}{\mathrm{d}t} = -\frac{F_2}{\tau_{\text{cav}2}} + \kappa_2 F_2 \Delta N_2 ,
\]

\[
\frac{\mathrm{d}\Delta N_2}{\mathrm{d}t} = \frac{1}{\tau} (\Delta N_{02} - \Delta N_2) - 2^* \kappa_2 F_2 \Delta N_2 ,
\]

where we have supposed that the two modes may have two different photon lifetimes and exhibit two different laser cross sections and pumping rates. Equations (7.1,7.2) and (7.3,7.4) constitute two completely independent sets of equations. If we want to take into account the possible competition of the two modes for gain, we can suppose that each population inversion $\Delta N_i$ with $i = 1, 2$ is subjected not only to self-saturation terms such as $2^* \kappa_i F_i \Delta N_i$ but also to cross-saturation terms $2^* \kappa_i \xi_{ij} F_j \Delta N_i$, leading to:
7.2. MODE COMPETITION

\[
\begin{align*}
\frac{dF_1}{dt} &= -\frac{F_1}{\tau_{cav1}} + \kappa_1 F_1 \Delta N_1, \quad (7.5) \\
\frac{d}{dt} \Delta N_1 &= \frac{1}{\tau} (\Delta N_{01} - \Delta N_1) - 2^* \kappa_1 \Delta N_1 (F_1 + \xi_{12} F_2), \quad (7.6) \\
\frac{dF_2}{dt} &= -\frac{F_2}{\tau_{cav2}} + \kappa_2 F_2 \Delta N_2, \quad (7.7) \\
\frac{d}{dt} \Delta N_2 &= \frac{1}{\tau} (\Delta N_{02} - \Delta N_2) - 2^* \kappa_2 \Delta N_2 (F_2 + \xi_{21} F_1). \quad (7.8)
\end{align*}
\]

The (positive) ratios $\xi_{12}$ and $\xi_{12}$ of the cross- to self-saturation coefficients that we have introduced here can take any value. Let us repeat one more time that this is the simplest way to introduce mode competition and that in real situations, extra terms must also usually be added to equations (7.5) and (7.7).

7.2 Mode competition in class-A lasers

If we consider a class-A laser, equations (7.6) and (7.8) can be adiabatically eliminated, leading to:

\[
\begin{align*}
\Delta N_1 &= \Delta N_{01} \frac{1}{1 + (F_1 + \xi_{12} F_2)/F_{sat1}}, \quad (7.9) \\
\Delta N_2 &= \Delta N_{02} \frac{1}{1 + (F_2 + \xi_{21} F_1)/F_{sat2}}, \quad (7.10)
\end{align*}
\]

with

\[
\begin{align*}
F_{sat1} &= \frac{1}{2^* \tau \kappa_1}, \quad (7.11) \\
F_{sat2} &= \frac{1}{2^* \tau \kappa_2}. \quad (7.12)
\end{align*}
\]

The laser behavior is then governed by the two remaining equations:

\[
\begin{align*}
\frac{dF_1}{dt} &= \frac{F_1}{\tau_{cav1}} \left[-1 + \frac{r_1}{1 + (F_1 + \xi_{12} F_2)/F_{sat1}}\right], \quad (7.13) \\
\frac{dF_2}{dt} &= \frac{F_2}{\tau_{cav2}} \left[-1 + \frac{r_2}{1 + (F_2 + \xi_{21} F_1)/F_{sat2}}\right]. \quad (7.14)
\end{align*}
\]
where we have introduced the relative excitation ratios \( r_i = \frac{\tau_{\text{cav},i}}{\kappa_i} \Delta N_{0i} \) for the two modes. We suppose here that the two modes are above threshold, meaning that \( r_1 > 1 \) and \( r_2 > 1 \). This makes the steady-state solution \( F_1 = F_2 = 0 \) unstable, leaving us with three possible steady-state solutions.

### 7.2.1 Simultaneous oscillation of the two modes

Let us consider the steady-state solution of equations (7.13) and (7.14) in which \( F_1 \neq 0 \) and \( F_2 \neq 0 \). This solution is:

\[
F_1 = F_{\text{sat}1}(r_1 - 1) - \xi_{12} F_{\text{sat}2}(r_2 - 1) \left/ \left( 1 - \xi_{12} \xi_{21} \right) \right.,
\]

(7.15)

\[
F_2 = F_{\text{sat}2}(r_2 - 1) - \xi_{21} F_{\text{sat}1}(r_1 - 1) \left/ \left( 1 - \xi_{12} \xi_{21} \right) \right.,
\]

(7.16)

Of course, in the absence of competition (\( \xi_{12} = \xi_{21} = 0 \)), one gets the usual solution of a single-frequency laser above threshold (see equation (3.6)). To study the stability of this solution, we perform a linear stability analysis by writing

\[
F_1(t) = F_1^0 + f_1(t),
\]

(7.17)

\[
F_2(t) = F_2^0 + f_2(t),
\]

(7.18)

with \( |f_1(t)| \ll F_1^0 \) and \( |f_2(t)| \ll F_2^0 \). By injecting equations (7.17) and (7.18) into equations (7.13) and (7.14) with the help of equations (7.15) and (7.16) and keeping only first-order terms in \( f_1 \) and \( f_2 \) one obtains:

\[
\frac{df_1}{dt} = -\frac{F_1^0}{\tau_{\text{cav}1}} \frac{f_1 + \xi_{12} f_2}{r_1 F_{\text{sat}1}},
\]

(7.19)

\[
\frac{df_2}{dt} = -\frac{F_2^0}{\tau_{\text{cav}2}} \frac{f_2 + \xi_{21} f_1}{r_2 F_{\text{sat}2}},
\]

(7.20)

which can be rewritten:

\[
\frac{d}{dt} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = M \begin{pmatrix} f_1 \\ f_2 \end{pmatrix},
\]

(7.21)

The matrix \( M \) is given by

\[
M = - \begin{pmatrix} \frac{F_1^0}{\tau_{\text{cav}1} r_1 F_{\text{sat}1}} & \frac{\xi_{12} F_1^0}{\tau_{\text{cav}1} r_1 F_{\text{sat}1}} \\ \frac{F_2^0}{\tau_{\text{cav}2} r_2 F_{\text{sat}2}} & \frac{\xi_{21} F_2^0}{\tau_{\text{cav}2} r_2 F_{\text{sat}2}} \end{pmatrix}.
\]

(7.22)

The eigenvalues of \(-M\) are given by the characteristic equation:

\[
\left( \lambda - \frac{F_1^0}{r_1 \tau_{\text{cav}1} F_{\text{sat}1}} \right) \left( \lambda - \frac{F_2^0}{r_2 \tau_{\text{cav}2} F_{\text{sat}2}} \right) - \frac{\xi_{12} \xi_{21} F_1^0 F_2^0}{r_1 r_2 \tau_{\text{cav}1} \tau_{\text{cav}2} F_{\text{sat}1} F_{\text{sat}2}} = 0,
\]

(7.23)
leading to
\[ \lambda^2 - \lambda \left( \frac{F_0^1}{r_1 \tau_{cav1} F_{sat1}} + \frac{F_0^2}{r_2 \tau_{cav2} F_{sat2}} \right) + \frac{F_0^1 F_0^2}{r_1 r_2 \tau_{cav1} \tau_{cav2} F_{sat1} F_{sat2}} (1 - \xi_{12} \xi_{21}) = 0 . \] (7.24)

The solution given by equations (7.15) and (7.16) is stable provided the two solutions \( \lambda_+ \) and \( \lambda_- \) of equation (7.24) have positive real parts. Let us rewrite equation (7.24) in the following form:
\[ \lambda^2 - \lambda (\alpha + \beta) + \alpha \beta (1 - C) = 0 , \] (7.25)
where we have defined
\[ \alpha = \frac{F_0^1}{r_1 \tau_{cav1} F_{sat1}} , \] (7.26)
\[ \beta = \frac{F_0^2}{r_2 \tau_{cav2} F_{sat2}} , \] (7.27)
\[ C = \xi_{12} \xi_{21} . \] (7.28)

The discriminant of equation (7.25) is
\[ \Delta = (\alpha + \beta)^2 - 4\alpha \beta (1 - C) = (\alpha - \beta)^2 + 4\alpha \beta C . \] (7.29)

### 7.2.1.1 Case where \( \alpha > 0 \) and \( \beta > 0 \)

In this case, one can see that the sum \( \lambda_+ + \lambda_- = \alpha + \beta \) is positive. Since \( \lambda_+ \lambda_- = \alpha \beta (1 - C) \), one can see that \( \lambda_+ \) and \( \lambda_- \) are both positive only if \( C < 1 \). In the case where \( C > 1 \), the simultaneous oscillation of the two modes is unstable.

### 7.2.1.2 Case where \( \alpha > 0 \) and \( \beta < 0 \) with \( C < 1 \)

In this case, equation (7.29) leads to:
\[ \Delta = \alpha^2 + \beta^2 + 2|\alpha \beta| - 4C|\alpha \beta| > \alpha^2 + \beta^2 + 2|\alpha \beta| - 4|\alpha \beta| > 0 . \] (7.30)

Then \( \lambda_+ \) and \( \lambda_- \) and are both real. The stability depends on the sign of
\[ \lambda_- = \frac{1}{2} \left\{ \left( \alpha + \beta \right) - \sqrt{(\alpha + \beta)^2 - 4\alpha \beta (1 - C)} \right\} , \] (7.31)
which is clearly negative. The simultaneous oscillation of the two modes is unstable.
In conclusion, the solution given by equations (7.15) and (7.16), which corresponds to the simultaneous oscillation of the two modes, can be stable only when the so-called coupling constant $C$ verifies:

$$C = \xi_{12}\xi_{21} \leq 1.$$  \hspace{1cm} (7.32)

It is worth noticing that equation (7.32) is a necessary but not a sufficient condition to achieve simultaneous oscillation of the two modes. Indeed, a situation in which $F_1^0 \leq 0$, $F_2^0 \geq 0$ and $C \leq 1$ leads to the stable oscillation of mode 2 only. Indeed, in this case, the existence of mode 2 inhibits the oscillation of mode 1, as we are going to see now.

### 7.2.2 Oscillation of one mode only

Let us consider the following steady-state solution of equations (7.13) and (7.14):

$$F_1^0 = F_{sat1}(r_1 - 1),$$  \hspace{1cm} (7.33)

$$F_2^0 = 0,$$  \hspace{1cm} (7.34)

which corresponds to the oscillation of mode 1 only. To evaluate the stability of this solution, we rewrite $F_1(t)$ and $F_2(t)$ according to:

$$F_1(t) = F_1^0 + f_1(t),$$  \hspace{1cm} (7.35)

$$F_2(t) = f_2(t).$$  \hspace{1cm} (7.36)

By injecting equations (7.35) and (7.36) into equations (7.13) and (7.14) with the help of equations (7.33) and (7.34) and keeping only first-order terms in $f_1$ and $f_2$, one obtains:

$$\frac{df_1}{dt} = -\frac{F_1^0}{\tau_{cav1}} \frac{f_1 + \xi_{12}f_2}{r_1F_{sat1}},$$  \hspace{1cm} \hspace{1cm} \hspace{1cm} (7.37)

$$\frac{df_2}{dt} = \frac{f_2}{\tau_{cav2}} \left(-1 + \frac{r_2}{1 + \xi_{21}F_1^0/F_{sat2}}\right).$$  \hspace{1cm} (7.38)

We clearly see that the solution corresponding to the oscillation of mode 1 only is stable if

$$\frac{r_2}{1 + \xi_{21}F_{sat1}(r_1 - 1)/F_{sat2}} \leq 1,$$ \hspace{1cm} (7.39)

meaning that the gain of mode 2 saturated by the presence of mode 1 is lower than the losses, restraining mode 2 from reaching threshold.
7.3. TRANSIENT BEHAVIOR

Similarly, the solution corresponding to the oscillation of mode 2 only (with a number of photons $F_{\text{sat}2}(r_2 - 1)$) is stable if

$$\frac{r_1}{1 + \xi_{12}F_{\text{sat}2}(r_2 - 1)/F_{\text{sat}1}} \leq 1,$$

meaning that the gain of mode 1 saturated by the presence of mode 2 is lower than the losses, restraining mode 1 from reaching threshold.

7.2.3 Bistability

From equations (7.39) and (7.40), we can see that the laser can be in a bistable regime, i. e., exhibit the two following stable steady-state solutions:

$$F_1 = F_{\text{sat}1}(r_1 - 1) \text{ and } F_2 = 0,$$

$$F_1 = 0 \text{ and } F_2 = F_{\text{sat}2}(r_2 - 1),$$

if the conditions (7.39) and (7.40) are simultaneously fulfilled. It is easy to show that the simultaneous fulfilment of these two conditions leads to:

$$C = \xi_{12}\xi_{21} \geq 1.$$  \hspace{1cm} (7.43)

7.3 Transient behavior of a two-frequency class-B laser

Let us now consider the case of a class-B two-mode laser, in which the adiabatic elimination of the population inversion is not possible (see chapter 2). Some interesting new physics occurs if we consider the case $C \leq 1$ in which the two modes may oscillate simultaneously. To describe the transient behavior of this laser, we must consider equations (7.5-7.8) which then admit the following steady-state solution:

$$\Delta N_1^0 = \frac{1}{\kappa_1 \tau_{\text{cav}1}},$$

$$\Delta N_2^0 = \frac{1}{\kappa_2 \tau_{\text{cav}2}},$$

$$F_1^0 = \frac{F_{\text{sat}1}(r_1 - 1) - \xi_{12}F_{\text{sat}2}(r_2 - 1)}{1 - \xi_{12}\xi_{21}},$$

$$F_2^0 = \frac{F_{\text{sat}2}(r_2 - 1) - \xi_{21}F_{\text{sat}1}(r_1 - 1)}{1 - \xi_{12}\xi_{21}}.$$
To find the transient behavior of this system when it is close to its steady-state solution, one must in general introduce the deviation with respect to the steady-state solution under the form of a 4-vector. Then, after linearization, one obtains four coupled linear ordinary differential equations, leading to a fourth-order characteristic polynomial and to very cumbersome calculations. To simplify this problem, let us consider that the two modes play symmetrical roles, i.e., that they have the same gains, losses, etc...

\[ \tau_{\text{cav}1} = \tau_{\text{cav}2} \equiv \tau_{\text{cav}} , \]  
\[ \kappa_1 = \kappa_2 \equiv \kappa , \]  
\[ \Delta N_{01} = \Delta N_{02} \equiv \Delta N_0 , \]  
\[ \xi_{12} = \xi_{21} \equiv \xi , \]

leading to:

\[ F_{\text{sat}1} = F_{\text{sat}2} \equiv F_{\text{sat}} , \]
\[ r_1 = r_2 \equiv r , \]
\[ \Delta N^0_1 = \Delta N^0_2 = \frac{1}{\kappa \tau_{\text{cav}}} \equiv \Delta N_{\text{th}} , \]
\[ F^0_1 = F^0_2 = \frac{F_{\text{sat}}(r - 1)}{1 + \xi} \equiv F^0 . \]

The symmetry of the system now encourages us to consider two different types of relaxation oscillations. Indeed, like in the case of two coupled identical mechanical oscillators, we can expect the system to exhibit two different oscillation modes: one oscillation mode in which the intensities of the two modes are in phase and one oscillation mode in which the two modes are in antiphase.

### 7.3.1 Standard relaxation oscillations

Let us first suppose that the system has an oscillation mode in which the two modes have the same behavior, namely

\[ F_1(t) = F_2(t) = F^0[1 + x(t)] , \]
\[ \Delta N_1(t) = \Delta N_2(t) = \Delta N_{\text{th}}[1 + y(t)] . \]

We inject equations (7.56) and (7.57) into equations (7.5-7.8). After having taken equations (7.48-7.55) into account, we obtain the following set of ordinary differential equations:

\[ \frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = M \begin{pmatrix} x \\ y \end{pmatrix} , \]
7.3. TRANSIENT BEHAVIOR

with the matrix \( M \) given by:

\[
M = - \begin{pmatrix}
0 & 1/\tau_{\text{cav}} \\
-(r-1)/\tau & -r/\tau
\end{pmatrix}.
\] (7.59)

The eigenvalues of this matrix are the same as those obtained in the case of a single-frequency laser studied in section 5.1:

\[
\lambda_{\pm} = -\frac{r}{2\tau} \pm \frac{i}{2} \sqrt{4(r-1)\tau_{\text{cav}} - \left(\frac{r}{\tau}\right)^2},
\] (7.60)

leading to the standard expression of the relaxation oscillation frequency in the case where \( \tau_{\text{cav}} \ll \tau \):

\[
f_{\text{relax}} \simeq \frac{1}{2\pi} \sqrt{\frac{r-1}{\tau_{\text{cav}}}}.
\] (7.61)

7.3.2 Antiphase relaxation oscillations

Let us now consider the possibility for the system to exhibit an oscillation mode in which the two modes have exactly opposite behaviors, namely

\[
F_1(t) = F_0[1 + x(t)],
\] (7.62)

\[
F_2(t) = F_0[1 - x(t)],
\] (7.63)

\[
\Delta N_1(t) = \Delta N_{\text{th}}[1 + y(t)],
\] (7.64)

\[
\Delta N_2(t) = \Delta N_{\text{th}}[1 - y(t)].
\] (7.65)

We inject equations (7.62-7.65) into equations (7.13) and (7.14). After having taken equations (7.48-7.55) into account, we are left with the following set of ordinary differential equations:

\[
\frac{dy}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = M \begin{pmatrix} x \\ y \end{pmatrix},
\] (7.66)

with the matrix \( M \) now given by:

\[
M = - \begin{pmatrix}
0 & 1/\tau_{\text{cav}} \\
-(1-x)\tau_{\text{cav}} & -r/\tau
\end{pmatrix}.
\] (7.67)

The eigenvalues of this matrix are now given by:

\[
\lambda_{\pm} = -\frac{r}{2\tau} \pm \frac{i}{2} \sqrt{4\left(\frac{1-x}{1+x}\right)\left(\frac{r-1}{\tau_{\text{cav}}}\right) - \left(\frac{r}{\tau}\right)^2}.
\] (7.68)
In the case of a class-B laser \((\tau_{\text{cav}} \ll \tau)\), this corresponds to the following antiphase frequency:

\[
\begin{align*}
\rightarrow f_{\text{anti}} \approx & \frac{1}{2\pi} \sqrt{\left(\frac{1 - \xi}{1 + \xi}\right) \left(\frac{r - 1}{\tau \tau_{\text{cav}}}\right)}.
\end{align*}
\]

This frequency of the antiphase oscillations is related to the standard relaxation oscillation frequency of equation (7.61) by the following relation:

\[
\begin{align*}
\frac{f_{\text{anti}}}{f_{\text{relax}}} = & \frac{1 - \xi}{1 + \xi} = \frac{1 - \sqrt{C}}{1 + \sqrt{C}}.
\end{align*}
\]

The existence of these two different relaxation frequencies is illustrated in figure 7.1. This figure has been obtained for an Er,Yb:Glass laser sustaining the oscillation of two linearly polarized modes aligned along the \(x\) and \(y\) directions (\(z\) corresponds to the laser propagation axis). Figure 7.1(a) displays the evolution of the intensities of the two modes versus time. The relaxation oscillation frequencies are excited by the laser noise, as we have seen in chapter 6. However, here, it is clear that these evolutions contain two frequencies: a fast frequency for which the fluctuations of the intensities of the two modes are in phase and a slow frequency for which the two intensities are in antiphase. This is even clearer in the spectra of Figure 7.1(b): The antiphase peak is clearly visible when one detects one of the two modes only and disappears when the two intensities are added. From the ratio of the frequencies of these two peaks, one can deduce the value of the coupling constant \(C\) using equation (7.70).

### 7.4 Injection locking

We now turn to a completely different topic corresponding to the situation of Figure 7.2. We consider two lasers (with their two frequencies) in interaction. More precisely, we consider a single-frequency class-A laser (the so-called slave laser) based on a unidirectional ring cavity, which is injected by a laser (the so-called master laser).

#### 7.4.1 Equations of the injected laser

We suppose that the master laser oscillates at frequency \(\omega_1\) and we develop the intracavity field of the slave laser at the frequency \(\omega_1\) of the master laser:

\[
E(z, t) = A(t)e^{-i(\omega_1 t - k z)} + \text{c.c.} .
\]
We also suppose that the electric field of the injected field incident on mirror $M_1$ is $E_1$ with a complex amplitude $A_1$. In these conditions, the equation of evolution of the complex amplitude of the intracavity field is:

$$\frac{dA}{dt} = -\frac{A}{2\tau_{cav}} \left[ 1 - \frac{r}{1 + \frac{2\pi n_0 A^2}{I_{sat}}} \right] + i\delta_{cav} A + \frac{1}{2\tau_{inj}} A_1, \quad (7.72)$$

with

$$\delta_{cav} = \omega_1 - \omega_q, \quad (7.73)$$

where $\omega_q$ is the free-running laser frequency and with

$$\tau_{inj} = \frac{L_{cav, opt}}{c_0} \frac{1}{T_1}. \quad (7.74)$$
where $T_1$ is the transmission of mirror $M_1$ through which the master laser beam is injected.

Let us now introduce the amplitudes and phases of the two fields:

$$\mathbf{A} = |A|e^{i\phi}, \quad (7.75)$$

$$A_1 = |A_1|e^{i\phi_1}. \quad (7.76)$$

Then, the left-hand side of equation (7.72) reads:

$$\frac{dA}{dt} = e^{i\phi} \left( \frac{d|A|}{dt} + i\dot{\phi}|A| \right). \quad (7.77)$$

By multiplying the left-hand and right-hand sides of equation (7.72) by $e^{-i\phi}$ and taking the real and imaginary parts of the resulting equation one gets:

$$\frac{d|A|}{dt} = -\frac{|A|}{2\tau_{cav}} \left[ 1 - \frac{r}{1 + \frac{2\epsilon}{\epsilon_0 |A|^2}} \right] + \frac{1}{2\tau_{inj}} |A_1| \cos(\phi_1 - \phi), \quad (7.78)$$

$$\frac{d\phi}{dt} = \omega_1 - \omega_q + \frac{1}{2\tau_{inj}} \frac{|A_1|}{|A|} \sin(\phi_1 - \phi). \quad (7.79)$$

### 7.4.2 Behavior in the case of a weak injection

For simplicity, let us now suppose that the master laser field amplitude $A_1$ is weak, allowing us to neglect the injected term in equation (7.78). Then
the steady-state intracavity field (real) amplitude $A_0$ is given by the usual equality between the saturated gain and the losses:

$$1 - \frac{r}{1 + \frac{2\omega_0 A_0^2}{I_{\text{sat}}}} = 0 ,$$

(7.80)

leading to

$$A_0^2 = \frac{I_{\text{sat}}}{2\omega_0 \varepsilon_0} (r - 1) .$$

(7.81)

Equation (7.79) then becomes:

$$\frac{d\varphi}{dt} = \omega_1 - \omega_q + \frac{1}{2\tau_{\text{inj}}} \frac{|A_1|}{A_0} \sin(\varphi_1 - \varphi) .$$

(7.82)

If we suppose that $\varphi_1 = 0$ and introduce the lock-in frequency:

$$\omega_L = \frac{1}{2\tau_{\text{inj}}} \frac{|A_1|}{A_0} ,$$

(7.83)

we can see that equation (7.82) reads:

$$\frac{d\varphi}{dt} = \omega_1 - \omega_q - \omega_L \sin(\varphi) .$$

(7.84)

This differential equation is the so-called Adler’s equation. It has been introduced by Adler to describe the behavior of coupled nonlinear electronic oscillators. However, is is ubiquitous in physical systems involving frequency locking between several oscillators such as lasers, mechanical oscillators, gyroscopes, etc...

### 7.4.2.1 Locked solution

In the case where

$$|\omega_1 - \omega_q| < \omega_L ,$$

(7.85)

equation (7.84) has the following steady-state solution:

$$\sin(\varphi) = \frac{\omega_1 - \omega_q}{\omega_L} .$$

(7.86)

In this case, one can see that the slave laser oscillates at the frequency of the master laser: the slave laser is locked to the master laser. Or, in other words, the master laser has transferred its frequency to the slave laser. This is of

---

1R. Adler, *Proc. IEEE* 61, 1380 (1973)
course possible as long as equation (7.86) leads to a value of \( \sin(\varphi) \) between -1 and 1, i.e., as long as the detuning \( |\omega_1 - \omega_q| \) between the master and the slave lasers is smaller than the locking region \( \pm \omega_L \). According to equation (7.83), the width of this locking region is proportional to the square root of the master laser intensity.

### 7.4.2.2 Unlocked solution

In the case where

\[
|\omega_1 - \omega_q| > \omega_L ,
\]

\[
\text{(7.87)}
\]
7.4. INJECTION LOCKING

equation (7.84) can no longer have any steady-state solution. The phase
difference $\varphi$ between the slave and the master laser then evolves versus time,
leading to a beat note frequency between the two lasers which can be deduced
by re-writing equation (7.84) in the following form:

$$\frac{d\varphi}{\omega_1 - \omega_q - \omega_L \sin(\varphi)} = dt,$$

i. e.

$$\frac{d\varphi}{1 - \frac{\omega_L}{\omega_1 - \omega_q} \sin(\varphi)} = (\omega_1 - \omega_q)dt. \quad (7.89)$$

By integration over one oscillation period $T$, i. e., for $\varphi$ varying from 0 to
$2\pi$, we get:

$$\int_0^{2\pi} \frac{d\varphi}{1 - \frac{\omega_L}{\omega_1 - \omega_q} \sin(\varphi)} = (\omega_1 - \omega_q)T. \quad (7.90)$$

Using the fact that, for $|a| < 1$,

$$\int_0^{2\pi} \frac{d\varphi}{1 - a \sin(\varphi)} = \frac{2\pi}{\sqrt{1 - a^2}}, \quad (7.91)$$

we obtain the following expression for the beatnote period $T$ between the
two lasers:

$$T = \frac{2\pi}{\sqrt{(\omega_1 - \omega_q)^2 - \omega_L^2}}, \quad (7.92)$$

which corresponds to the following value of the beatnote frequency between
the two lasers:

$$f_{\text{beat}} = \frac{1}{2\pi} \sqrt{\frac{(\omega_1 - \omega_q)^2 - \omega_L^2}{\omega_1 - \omega_q}}. \quad (7.93)$$

The evolution of this beatnote frequency versus the detuning $\omega_1 - \omega_q$ is
shown in figure 7.3. One clearly sees the two (locked and unlocked) regions,
corresponding to the solutions of equations (7.86) and (7.93). In particular,
one can see that for $|\omega_1 - \omega_q| \gg \omega_L$, the slave laser oscillates at its free-
running frequency $\omega_q$, corresponding to the full line of figure 7.3. Figure 7.4
presents some typical solution of Adler’s equation (7.84) in three situations:
within the lock-in region, just at the output of the lock-in region, and well
above the lock-in threshold. In particular, one can see the anharmonic nature
of the beatnote when one is just at the outside of the lock-in region. When
one is well outside the lock-in region, the linear nature of the time evolution
of $\varphi$ corresponds to the frequency shift $\omega_1 - \omega_q$.

This injection-locking phenomenon has a lot of applications. In particu-
lar, it is used when one wants to build a powerful and spectrally pure laser. In
this case, one often uses the so-called MOPA (Master Oscillator Power Amplifier) architecture\(^2\) in which a well controlled low-power frequency stable laser is injected into a powerful but spectrally broad laser. Then, provided the powerful laser locks to the frequency of the master laser, one is able to transfer the spectral purity of the master laser to the powerful slave laser.

\(^2\)We refer here to the types of MOPA in which the so-called amplifier is actually a slave laser and not a simple optical amplifier.
Chapter 8

Mode-locked laser operation

We have seen in chapter 5 that a single-frequency class-B laser is able to emit short light pulses when its losses (or its gain) are modulated or switched. We have seen that in such a Q-switched laser, the pulse duration is of the order of the lifetime of the photons in the cavity, i.e., in the nanosecond range. However, many applications today require the use of much shorter pulses, i.e., in the picosecond or the femtosecond regime. This is clearly beyond the capabilities of Q-switched lasers and requires completely different strategies.

8.1 Introduction

Let us consider a short pulse of light of angular frequency $\omega_p$ and duration $\Delta t \gg 2\pi/\omega_p$. For the sake of simplicity, we suppose that this pulse has a Gaussian amplitude and exhibits no chirp ($\Delta t$ is real). Then its electric field at a given location reads:

$$E(t) = E_0 \exp \left[ -\frac{t^2}{2\Delta t^2} \right] \exp(-i\omega_p t) + c.c. \ . \quad (8.1)$$

We remind here the definition of the analytical signal, which is the positive frequency part of $E(t)$:

$$E^{(+)}(t) = E_0 \exp \left[-\frac{t^2}{2\Delta t^2} \right] \exp(-i\omega_p t) , \quad (8.2)$$

leading to:

$$E(t) = E^{(+)}(t) + [E^{(+)}(t)]^* = E^{(+)}(t) + E^{(-)}(t) \ . \quad (8.3)$$
We then calculate the Fourier transform of $E^+(t)$, which is the positive frequency part of the Fourier transform of $E(t)$:

$$
\tilde{E}^+(\omega) = \tilde{E}_0 \exp \left[ -\frac{(\omega - \omega_p)^2}{2\Delta \omega^2} \right],
$$

(8.4)

with

$$
\Delta t \Delta \omega = 1.
$$

(8.5)

Suppose for example that one wants to generate a pulse of duration $\Delta t = 1 \text{ ps}$. The laser spectrum must consequently be broader than $\Delta \omega/2\pi \simeq 150 \text{ GHz}$. This simple example leads to the two following important consequences. First, one can see that short pulses can only be obtained from active media exhibiting broadband gain. This includes doped glasses (Nd:glass), dyes, semiconductors, vibronic lasers (Ti:Sapphire), etc... Second, lasers emitting short pulses must exhibit a very large number of oscillating modes, contrary to the Q-switched laser considered in chapter 5. For example, if one wants to make a laser with a 1-m long ring cavity ($c_0/L_{\text{cav}} = 300 \text{ MHz}$) emit 1-ps long pulses, the number of longitudinal modes must be of the order of $150 \text{ GHz}/300 \text{ MHz} = 500$. This is illustrated in figure 8.1 where one can see the sum of $N$ successive longitudinal modes leads to pulse widths of the order of one $N^{\text{th}}$ of the round-trip time in the laser cavity. Moreover, since all the modes have frequencies separated by the free spectral range of the cavity, the pulse repetition rate is given by the cavity free spectral range. In the time domain, one can imagine the superposition of the $N$ modes to give rise to a pulse propagating in the cavity at the velocity of light $c_0$, hence leading to a repetition period given by the time the pulse takes to make one round-trip inside the cavity, i.e., $L_{\text{cav}}/c_0$.

One can also notice that, in figure 8.1, all the cosine waves start with the same phase. This phase locking condition is important for the laser to generate pulses. Let us indeed consider the sum of the fields of $N$ successive modes of equal real amplitudes $E_0$:

$$
E(t) = E_0 \sum_{n=0}^{N-1} e^{i\varphi_n} e^{-i[(\omega_0+n\Delta)t]} + \text{c.c.},
$$

(8.6)

where the $\varphi_n$’s are the phases of the successive modes, $\Delta/2\pi$ is the free spectral range of the cavity and $\omega_0$ is a given frequency offset. If the $\varphi_n$’s are
all equal to a given $\varphi$, then equation (8.6) becomes:

$$E(t) = E_0 e^{i\varphi} e^{-i\omega_0 t} \sum_{n=0}^{N-1} e^{-in\Delta t} + c.c.$$ 

$$= E_0 e^{i\varphi} e^{-i\omega_0 t} \frac{1 - e^{-iN\Delta t}}{1 - e^{-i\Delta t}} + c.c.$$ 

$$= E_0 \frac{\sin \frac{N\Delta t}{2}}{\sin \frac{\Delta t}{2}} e^{i\varphi} e^{-i[\omega_0 + (N-1)\frac{\Delta}{2}]} + c.c. \quad (8.7)$$

The intensity of the field of equation (8.7) is given by

$$I(t) = I_0 \left( \frac{\sin \frac{N\Delta t}{2}}{\sin \frac{\Delta t}{2}} \right)^2,$$  

where $I_0$ is the intensity of each frequency in the sum of equation (8.4). Equation (8.8) corresponds to pulses having a width of the order of $\frac{2\pi}{N\Delta}$ and a peak intensity equal to $N^2$ times the intensity of each mode, as shown in figure 8.2(a).
8. MODE-LOCKING

If now the phases $\varphi_n$ of the successive modes take random values, the evolution of the light intensity versus time behaves like in figure 8.2(b). One can notice that, even if the intensity is periodic with a period given by the intracavity round-trip time $2\pi/\Delta$, it does no longer consist in a train of pulses as in figure 8.2(a). In particular, the maximum intensity is much lower.

It is thus clear that if we want to generate short and powerful pulses, we need our laser to generate as many longitudinal modes as possible, but with a well chosen phase relationship between them. This is the so-called **mode-locking** behavior. This mode-locking regime is, to some extent, similar to the injection locking described in section 7.4, but for many different modes. It can be reached either using the intrinsic nonlinearities of the laser (**passive mode-locking**) or by introducing a modulation of the losses inside the cavity (**active mode-locking**). In this chapter, we will deal mainly with the latter technique, first by deriving the equations of evolutions of the laser modes (see section 8.2) and then by describing the intracavity evolution of the laser pulse using a so-called temporal approach (see sections 8.3 and 8.4).

### 8.2 Spectral approach to active mode-locking

Let us consider the unidirectional ring laser of Figure 8.3. Its cavity contains an active medium and an intensity modulator (for example, an acousto-optic or electro-optic modulator). The transmission of this modulator is sinusoidally varied at an angular frequency $\omega_M$. The frequency $\omega_M/2\pi$ is chosen close to the round-trip frequency $c_0/L_{cav}$. In these conditions, we
8.2. SPECTRAL APPROACH

Hope that the 'opening' of this modulator will force the laser to sustain oscillation of an intracavity pulse that will pass through the modulator when its intensity is maximum. The intensity transmission of the modulator can be written:

$$\Theta(t) = \Theta_0 \left(1 - \mu \sin^2 \frac{\omega_M t}{2}\right). \quad (8.9)$$

An example of such a modulated transmission is shown in figure 8.4. If we suppose that $\mu \ll 1$ and that the modulator exhibits no chirp, then its amplitude transmission reads:

$$\theta(t) = \theta_0 \left(1 - \frac{\mu}{2} \sin^2 \frac{\omega_M t}{2}\right), \quad (8.10)$$

with $\Theta_0 = \theta_0^2$ (we suppose that $\theta_0$ is real). Let us suppose that a monochromatic traveling plane wave is incident on this modulator located at $z = 0$:

$$E_{\text{in}}(z, t) = A_{\text{in}} e^{-i(\omega t - k z)} + c.c., \quad (8.11)$$

then, since equation (8.10) can be re-written as

$$\theta(t) = \theta_0 \left(1 - \frac{\mu}{4} + \frac{\mu}{8} e^{i\omega_M t} + \frac{\mu}{8} e^{-i\omega_M t}\right), \quad (8.12)$$

the field at the output of the modulator reads:

$$E_{\text{out}}(0, t) = \theta_0 \left(1 - \frac{\mu}{4}\right) A_{\text{in}} e^{-i\omega t} + \theta_0 \frac{\mu}{8} A_{\text{in}} e^{-i(\omega - \omega_M) t} + \theta_0 \frac{\mu}{8} A_{\text{in}} e^{-i(\omega + \omega_M) t} + c.c., \quad (8.13)$$
Figure 8.4: Time evolution of the transmission of the modulator for $\mu = 0.1$.

where we have neglected the phase accumulated through propagation in the modulator. Equation (8.13) thus shows that the modulator creates two side bands on the incident light spectrum, located at $\pm \omega_M$ of the incident frequency, as shown in figure 8.5.

Figure 8.5: Light spectrum before and after the modulator.

Let us now suppose that the laser intracavity field is multimode:

$$E(z, t) = \sum_n A_n(t)e^{-i(\omega_n t - k_n z)} + c.c. ,$$  \hspace{1cm} (8.14)

where we have developed the field on a set of frequencies $\{\omega_n\}$ that will be precised later. Then the equation of evolution of the complex amplitude $A_n$ reads:

$$\frac{dA_n}{dt} = -\frac{1}{2\tau'_{\text{cav}}} \left[ 1 - \frac{\Delta N_n}{\Delta N_{\text{th},n}} \right] A_n - i\delta_n A_n$$

$$+ \gamma A_{n-1}e^{-i(\omega_n - 1 - \omega_n + \omega_M)t} + \gamma A_{n+1}e^{-i(\omega_n + 1 - \omega_n - \omega_M)t} ,$$  \hspace{1cm} (8.15)
where
\[ \frac{1}{2\tau'_{\text{cav}}} = \frac{1}{2\tau_{\text{cav}}} + \frac{c_0}{L_{\text{cav}}} \left[ 1 - \theta_0 \left( 1 - \frac{\mu}{4} \right) \right], \tag{8.16} \]
\[ \gamma = \frac{c_0}{L_{\text{cav}}} \theta_0 \frac{\mu}{8}, \tag{8.17} \]
\( \delta_n \) is the difference between \( \omega_n \) and the “true” resonance angular frequency of mode \( n \), taking mode pulling effects into account, and \( \Delta N_n \) and \( \Delta N_{\text{th},n} \) are the population inversion and population inversion at threshold for mode \( n \), taking saturation by all the laser modes into account. The factor \( c_0/L_{\text{cav}} \) in equation (8.17) corresponds to the number of times the intracavity light crosses the modulator per second. Notice that \( \tau'_{\text{cav}} \) in equation (8.16) takes the losses introduced by the modulator into account. Equation (8.15) shows that the modulator injects a part of the fields of modes \( n-1 \) and \( n+1 \) into mode \( n \). We can thus expect that under some circumstances, the phases of all the modes get locked, like in the injection locking phenomenon studied in section 7.4. This is even clearer if one separates the equations of evolution of the amplitude and phase of mode \( n \). To do so, we define:
\[ A_n = |A_n| \exp(i\varphi_n). \tag{8.18} \]
We then take the real and imaginary parts of equation (8.15) multiplied by \( \exp(-i\varphi_n) \) with
\[ \omega_n = \Omega_q + (n - q)\omega_M, \tag{8.19} \]
where \( \Omega_q \) is a resonance angular frequency of the cavity located more or less in the middle of the laser spectrum, leading to:
\[ \frac{d|A_n|}{dt} = -\frac{1}{2\tau'_{\text{cav}}} \left[ 1 - \frac{\Delta N_n}{\Delta N_{\text{th},n}} \right] |A_n| + \gamma |A_{n-1}| \cos (\varphi_n - \varphi_{n-1}) + \gamma |A_{n+1}| \cos (\varphi_{n+1} - \varphi_n), \tag{8.20} \]
\[ \dot{\varphi}_n = -\delta_n - (n - q)\Delta \omega - \gamma \frac{|A_{n-1}|}{A_n} \sin (\varphi_n - \varphi_{n-1}) - \gamma \frac{|A_{n+1}|}{A_n} \sin (\varphi_n - \varphi_{n+1}), \tag{8.21} \]
with
\[ \Delta \omega = \Delta - \omega_M, \tag{8.22} \]
where \( \Delta/2\pi \) is the free spectral range of the cavity.

The set of equations (8.20) and (8.21) for \( n = 0 \ldots N-1 \) is usually impossible to solve analytically. However, it permits to put into evidence
the physics of active mode-locking from a spectral point of view. It shows indeed that the modulator is there to create side bands to each mode which are injected into the side modes, in order to lock the phases of all the laser modes. Equation (8.21) is again an Adler equation (see section (7.4)) and it is clear that, if \( \delta_n = 0 \), this mode-locking will be more efficient if \( \Delta \omega = 0 \), i.e., if the modulation frequency of the losses exactly matches the laser free spectral range.

In the present section, we have considered only the case of active mode-locking in which an intracavity modulators injects part of the field of a given mode into its two neighbors in the spectral domain. However, there also exists situations in which such a modulator is not necessary and in which the nonlinearities of the active medium are sufficient to lock the phases of the modes. Indeed, suppose for example that the lifetime of the population inversion is shorter than the inverse of the free spectral range. Then, the beat note between modes \( n \) and \( n+1 \) in the active medium will create a modulation of the population inversion. Then, for example, the field of mode \( n \) will be diffracted by this temporal grating and be partly injected into mode \( n-1 \), thus creating an injection term in the equation of evolution of \( A_{n-1} \), similarly to what we have seen in the case of active mode-locking. This explains why multimode lasers sometimes naturally exhibit passive mode-locking.

The fact that equations (8.20) and (8.21) are very cumbersome makes them hardly usable to predict the real characteristics of mode-locked lasers. This is the reason why we now turn to temporal approaches in which the time evolution of the intracavity pulse is directly considered.

### 8.3 Temporal approach to active mode-locking

#### I: Kuizenga and Siegman’s model

Let us again consider the unidirectional ring laser of figure 8.3. Kuizenga and Siegman’s temporal approach consists in following the evolution of a chirpless Gaussian light pulse inside this cavity and in writing that this pulse is identical to itself after one round-trip inside the cavity. Like in section 8.1, we suppose that the electric field of this pulse reads, just before entering the amplifier:

\[
E(t) = E_0 \exp \left( -\frac{t^2}{2\Delta \tau^2} \right) \exp(-i\omega_p t) + c.c. . \tag{8.23}
\]

Let us remind that the positive part of its spectrum is given by:

\[
\tilde{E}^+(\omega) = \tilde{E}_0 \exp \left[ -\frac{(\omega - \omega_p)^2}{2\Delta\omega^2} \right] , \tag{8.24}
\]
with
\[ \Delta t \Delta \omega = 1. \]  \hspace{1cm} (8.25)

### 8.3.1 Passage through the amplifier

Let us suppose that the gain medium of length \( L_a \) has a small-signal gain coefficient whose maximum lies at \( \omega_p \) and exhibits a quadratic dependence around this frequency:

\[ \alpha_0(\omega) = \alpha_0(\omega_p) \left[ 1 - \left( \frac{\omega - \omega_p}{\delta \omega_a} \right)^2 \right]. \]  \hspace{1cm} (8.26)

Then the intensity and field transmission coefficients \( T_a \) and \( t_a \) of the amplifier read:

\[ T_a(\omega) = t_a^2(\omega) = \exp \left\{ G_0 \left[ 1 - \left( \frac{\omega - \omega_p}{\delta \omega_a} \right)^2 \right] \right\}, \]  \hspace{1cm} (8.27)

with the amplifier small signal gain at resonance given by:

\[ G_0 = \alpha_0(\omega_p)L_a. \]  \hspace{1cm} (8.28)

Then the light pulse at the output of the amplifier is given by:

\[ \tilde{E}'(+) = t_a(\omega)\tilde{E}(+) = \tilde{E}_0 \exp \left\{ -\frac{(\omega - \omega_p)^2}{2\Delta \omega^2} + \frac{G_0}{2} \left[ 1 - \left( \frac{\omega - \omega_p}{\delta \omega_a} \right)^2 \right] \right\}. \]  \hspace{1cm} (8.29)

This is again a Gaussian spectrum of width \( \delta \omega' \) given by:

\[ \left( \frac{1}{\Delta \omega'} \right)^2 = \left( \frac{1}{\Delta \omega} \right)^2 + \frac{G_0}{\delta \omega_a^2}. \]  \hspace{1cm} (8.30)

Thanks to equation (8.25), we can deduce the duration \( \Delta t' \) of the pulse at the output of the amplifier:

\[ (\Delta t')^2 = \Delta t^2 + \frac{G_0}{\delta \omega_a^2}. \]  \hspace{1cm} (8.31)

Thus, it appears that the passage of the pulse through the gain medium leads to a **spectral narrowing** and thus to a **time broadening** of the pulse, as shown in figure 8.6. This is due to the fact that the amplifier behaves like a bandpass filter for the laser spectrum.
8. MODE-LOCKING

Figure 8.6: Spectral narrowing of the pulse due to its passage in through the gain medium.

8.3.2 Passage through the modulator

In the case where the modulator has no losses, its transmission is given by equation (8.9) with $\theta_0 = 1$:

$$\Theta(t) = \theta(t)^2 = 1 - \mu \sin^2 \frac{\omega_M t}{2}.$$  \hspace{1cm} (8.32)

We are interested here in pulse durations $\Delta t$ much shorter that the modulator period $2\pi/\omega_M$. Then, in the vicinity of its maximum transmission, i. e., around $t = 0$, equation (8.32) can be approximated by:

$$\Theta(t) = \theta(t)^2 \approx 1 - \mu \frac{\omega_M^2}{4} t^2 \approx \exp \left(-\mu \frac{\omega_M^2}{4} t^2\right).$$ \hspace{1cm} (8.33)

Consequently, at the output of the modulator, the pulse of equation (8.29) becomes:

$$E''(t) = \theta(t)E'(t) = E'_0 \exp \left[-\frac{t^2}{2(\Delta t')^2} - \mu \frac{\omega_M^2}{8} t^2\right] \exp(-i \omega_pt) + c.c.$$ \hspace{1cm} (8.34)

This is again a Gaussian pulse of duration $\tau''$ given by:

$$\frac{1}{(\Delta t'')^2} = \frac{1}{(\Delta t')^2} + \frac{\mu \omega_M^2}{4}.$$ \hspace{1cm} (8.35)

Thanks to equation (8.25), we can deduce the width $\Delta \omega''$ of the spectrum of the pulse at the output of the modulator:

$$(\Delta \omega'')^2 = (\Delta \omega')^2 + \mu \frac{\omega_M^2}{4}.$$ \hspace{1cm} (8.36)

Thus, it appears that the passage of the pulse through the modulator leads to a spectral broadening and thus to a time shortening of the pulse, as shown in figure 8.7. This is due to the fact that the modulator behaves like a time gate.
8.3. TEMPORAL APPROACH I

8.3.3 Evolution after one cavity round-trip

By combining equations (8.31) and (8.35), we can relate the pulse duration after \( n + 1 \) round-trips to the one after \( n \) round-trips through:

\[
\left( \frac{1}{\Delta t_{n+1}} \right)^2 = \frac{1}{(\Delta t_n)^2} + \frac{G_0}{\delta \omega^2} + \frac{\mu \omega_M^2}{4}.
\] (8.37)

Similarly, the field amplitudes must obey:

\[
|E_{0,n+1}|^2 = (1 - \Pi) \exp(G_0) |E_{0,n}|^2,
\] where \( \Pi \) are the round-trip losses.

8.3.4 Steady-state regime

In steady-state, the field after one round-trip must be unchanged, leading, according to equation (8.38):

\[
G_0 = \Pi,
\] (8.39)

which is simply the threshold condition. Besides, writing \( \Delta t_{n+1} = \Delta t_n \) in equation (8.37) leads to the following equation for the steady-state value \( \Delta t_0 \) for the pulse duration:

\[
\left( \frac{1}{\Delta t_0} \right)^2 = \frac{1}{(\Delta t_0)^2} + \frac{G_0}{\delta \omega^2} + \frac{\mu \omega_M^2}{4}.
\] (8.40)

The positive solution of this equation reads:

\[
\Delta t_0^2 = \frac{G_0}{2\delta \omega^2} \left[ \sqrt{1 + \frac{16\delta \omega^2}{\mu \omega_M^2 G_0}} - 1 \right].
\] (8.41)

Figure 8.7: Time compression of the pulse due to its passage in through the modulator.
In usual short pulse lasers, $\delta \omega_a/2\pi$ is of the order of 1 THz while $\omega_M/2\pi$ is of the order of 100 MHz. One thus has $\delta \omega_a/\omega_M \gg 1$ and $G_0$ and $\mu$ are of the order of 1. Equation (8.41) thus becomes:

$$\Delta t_0 \simeq \sqrt{\frac{2}{\delta \omega_a \omega_M}} \left( \frac{G_0}{\mu} \right)^{1/4}. \quad (8.42)$$

One can see that when the gain bandwidth $\delta \omega_a$ or the modulation depth $\mu$ tend to 0, the pulse duration tends to infinity: the pulsed regime is no longer stable.

For $\mu = 0.1$, $\delta \omega_a/2\pi = 6$ THz (20 nm at 1 $\mu$m), $\omega_M/2\pi = 100$ MHz (3-m long cavity), and $G_0 = 0.5$, we obtain $\Delta t_0 = 14$ ps.

### 8.3.5 Build-up time of the pulsed regime

Equation (8.37) can be re-written as:

$$\Delta t_{n+1}^2 = \frac{\Delta t_n^2 + G_0}{1 + \frac{\mu^2}{4\omega_M^2} \Delta t_n^4 + \frac{\mu G_0}{4 \delta \omega_a^2}} \simeq \Delta t_n^2 + \frac{G_0}{\delta \omega_a^2} - \frac{\mu^2}{4 \omega_M^2} \Delta t_n^4. \quad (8.43)$$

Using equation (8.42), we normalize it to $\Delta t_0^2$, leading to:

$$\left( \frac{\Delta t_{n+1}}{\Delta t_0} \right)^2 - \left( \frac{\Delta t_n}{\Delta t_0} \right)^2 = \sqrt{\frac{\mu G_0 \omega_M}{2 \delta \omega_a}} \left[ 1 - \left( \frac{\Delta t_n}{\Delta t_0} \right)^4 \right]. \quad (8.44)$$

Let us introduce the quantity

$$\zeta = \sqrt{\frac{\mu G_0 \omega_M}{2 \delta \omega_a}} \quad (8.45)$$

and the function

$$y(n) = \left( \frac{\Delta t_n}{\Delta t_0} \right)^2, \quad (8.46)$$

then equation (8.44) is the discretized version of the following differential equation:

$$\frac{dy}{dn} = \zeta \left[ 1 - y(n)^2 \right]. \quad (8.47)$$

This differential equation admits the following solution:

$$y(n) = \frac{\tanh \zeta n + y(0)}{1 + y(0) \tanh \zeta n}, \quad (8.48)$$
8.3. TEMPORAL APPROACH I

Figure 8.8: Evolution of the pulse duration normalized to its steady-state value versus the number of round-trips inside the cavity. The values of the parameters are \( \mu = 0.1, G_0 = 0.5, \omega_M/2\pi = 100 \text{ MHz}, \) and \( \delta \omega_a/2\pi = 6 \text{ THz}. \)

\( y(0) \) being the initial value of \( y. \) Figure 8.8 shows an example of evolution of the pulse duration versus number of cavity round-trips for the same parameters as before (leading to a steady-state pulse duration \( \Delta t_0 = 14 \text{ ps} \)) and with an initial pulse duration \( \Delta t(0) = 7 \text{ ns}. \) With these parameters, one has \( \zeta = 1.86 \times 10^{-6}. \)

The build-up time corresponds roughly to the number of cavity round-trips \( n_{\text{build-up}} \) such that \( \zeta n_{\text{build-up}} = 2 \) in equation (8.48). Using equation (8.45), we obtain:

\[
    n_{\text{build-up}} = \frac{4 \delta \omega_a}{\sqrt{\mu G_0 \omega_M}},
\]

leading to a build-up time given by:

\[
    T_{\text{build-up}} = \frac{L_{\text{cav}}}{c_0} n_{\text{build-up}} = \frac{8 \pi \delta \omega_a}{\sqrt{\mu G_0 \omega_M^2}} = \frac{4 N L_{\text{cav}}}{\sqrt{\mu G_0}} \frac{N L_{\text{cav}}}{c_0},
\]

where we have introduced the number \( N = \delta \omega_a/\omega_M \) of modes contained in the gain bandwidth. In the case of our example, we find \( T_{\text{build-up}} = 10.7 \text{ ms}. \) The interpretation of equation (8.50) is straightforward: the build-up time is, to a numerical factor of the order of 1, proportional to the number of modes that must be phase locked times the round-trip time. This indicates that the laser roughly needs one round-trip per mode to phase-lock its entire spectrum.
8.3.6 Need for a more elaborate model

The model developed in the present section relies on the following hypotheses:

- The laser amplification is supposed to be linear: there are no saturation effects.
- The spectrum of the laser pulse is centered on the gain maximum.
- The amplifier bandwidth is large compared with the pulse bandwidth ($\delta \omega_a \tau \gg 1$). This justifies the use of second-order developments (parabolic approximation).
- The pulses are synchronized with the maximum of the transmission of the modulator.
- The modulation period is large compared with the duration of the pulses ($\omega_M \tau \ll 1$). This also justifies a second-order development for the transmission of the modulator.

Under these conditions, we have seen that this model permits to predict the order of magnitude of the pulse duration. However, a different model must be derived if we want to take into account the following phenomena:

- the saturation of the amplifying medium, allowing to predict the intensity of the laser;
- the dispersion effects;
- the possible frequency shift between the modulator and the cavity round-trip frequency and its consequences;
- the nonlinear effects such as the phase auto-modulation and the Kerr effect;
- the stability of the pulsed regime in the presence of noise.

The model described in the following section permits to take some of these effects into account, at the cost of an increased complexity.
8.4 Temporal approach to active mode-locking II: Haus’s model

This model is a perturbative model in which the pulse undergoes infinitesimal changes at each cavity round trip. The electric field is again described, at a given location inside the cavity, by:

\[ E(t) = E^{(+)}(t) + \text{c.c.} = \mathcal{E}(t) \exp(-i\omega_p t) + \text{c.c.} \quad . \] (8.51)

Each optical component is described by a transfer operator \( T_i \) which acts on the field amplitude \( \mathcal{E}(t) \). The evolution of the pulse during a time \( T \) is described by the product of all the operators describing the elements crossed by the pulse during this time \( T \):

\[ \mathcal{E}(t + T) = \prod_i T_i \mathcal{E}(t) \quad . \] (8.52)

If we suppose that the evolution of the pulse during time \( T \) is infinitesimal, we can linearize equation (8.52):

\[ \mathcal{E}(t + T) = \mathcal{E}(t) + T \frac{d\mathcal{E}}{dt} \quad . \] (8.53)

We also take into account the fact that the action of each element on the pulse is infinitesimal:

\[ T_i = 1 + dT_i \quad , \] (8.54)

leading to:

\[ \prod_i T_i = 1 + \sum_i dT_i \quad . \] (8.55)

Equation (8.52) eventually reads:

\[ T \frac{d\mathcal{E}}{dt} = \left( \sum_i dT_i \right) \mathcal{E} \quad . \] (8.56)

Our aim now is to determine the infinitesimal operator \( dT_i \) for all the optical elements of our laser and to write the self-consistency of the field envelope after one cavity round-trip. Here again, we consider the case of the cavity of figure 8.3, but we also take intracavity dispersive elements into account.
8. MODE-LOCKING

8.4.1 Laser amplifier

If we want to take into account both the imaginary and real parts of the susceptibility of the gain medium, i.e., both the gain and its associated dispersion, the amplitude gain coefficient of the amplifier can be modeled as the following complex quantity:

$$\frac{\alpha(\omega)}{2} = \frac{\alpha(\omega_p)}{2} \left[ 1 - i \frac{\omega - \omega_p}{\delta\omega_a} \right]^{-1}, \quad (8.57)$$

where $\omega_p$ is the maximum gain frequency and $\delta\omega_a$ is the gain bandwidth. Indeed, if we model the gain spectrum by a simple Lorentzian, equations (1.49) and (1.50) show that the ratio of the real and imaginary parts of the susceptibility is $(\omega - \omega_p)/\delta\omega_a$, which is exactly the ratio of the dispersion and the gain contributions contained in equation (8.57). Then, for small detunings and small gain, the amplitude gain for one pass through the amplifier reads:

$$\exp \left[ \frac{\alpha(\omega)}{2} L_a \right] \simeq 1 + \frac{G_0}{2} \left[ 1 + i \frac{\omega - \omega_p}{\delta\omega_a} - \left( \frac{\omega - \omega_p}{\delta\omega_a} \right)^2 \right], \quad (8.58)$$

with

$$G_0 = \alpha(\omega_p) L_a. \quad (8.59)$$

Since the field amplitude $E(t)$ and the positive frequency part $\tilde{E}^{(+)}(\omega)$ of the Fourier transform of the field are related by

$$E(t) = \frac{1}{\sqrt{2\pi}} \int d\omega \tilde{E}^{(+)}(\omega) e^{-i(\omega - \omega_p)t}, \quad (8.60)$$

we have the following equivalence between the time derivative in the time domain and the product by the frequency in the frequency domain:

$$-i(\omega - \omega_p) \Leftrightarrow \frac{d}{dt}. \quad (8.61)$$

Using equation (8.61) in equation (8.58), we obtain:

$$dT_{\text{amp}} = \frac{G_0}{2} \left[ 1 - \frac{1}{\delta\omega_a} \frac{d}{dt} + \frac{1}{(\delta\omega_a)^2} \frac{d^2}{dt^2} \right]. \quad (8.62)$$

8.4.2 Modulator

Using equation (8.33), we obtain, when the pulse is in the vicinity of a transmission maximum,

$$dT_{\text{mod}} = -\frac{\mu}{8} \omega_M^2 t^2. \quad (8.63)$$
If now there is a frequency difference between the modulator frequency $\omega_M$ and the cavity frequency $\Delta$ given by:

$$\Delta \omega = \Delta - \omega_M$$  \hspace{1cm} (8.64)

The field and the modulation get shifted at each cavity round-trip by a time delay

$$\delta T_M = 2\pi \left( \frac{1}{\omega_M} - \frac{1}{\Delta} \right) \approx 2\pi \frac{\Delta \omega}{\Delta^2} .$$  \hspace{1cm} (8.65)

If $\delta T_M > 0$, the modulator works at a frequency smaller than the free spectral range, meaning that the field will be delayed with respect to the modulator. If this delay is small, we can take it into account at first order by writing:

$$dT_{\text{mod}} = -\frac{\mu}{8} \omega^2 t^2 - \delta T_M \frac{d}{dt} .$$  \hspace{1cm} (8.66)

### 8.4.3 Intracavity dispersion

If the cavity contains components whose refractive index varies with $\omega$, their optical length will depend on $\omega$. This will create a phase shift between the different spectral components of the pulse. Let us perform a Taylor development of the wavenumber:

$$k(\omega) = k(\omega_p) + k'(\omega_p) (\omega - \omega_p) + \frac{1}{2} k''(\omega_p) (\omega - \omega_p)^2 .$$  \hspace{1cm} (8.67)

Then, the propagation through a length $l$ of such a dispersive medium is described by:

$$e^{ik(\omega)l} = \exp \left\{ i \left[ k(\omega_p) + k'(\omega_p) (\omega - \omega_p) + \frac{1}{2} k''(\omega_p) (\omega - \omega_p)^2 \right] l \right\} .$$  \hspace{1cm} (8.68)

Then the phase shifts accumulated inside all the intracavity optical elements read:

$$\prod_i e^{i k_i(\omega) l_i} = \prod_i \exp [ i k_i(\omega_p) l_i ]$$

$$\times \prod_i \exp [ i k'_i(\omega_p) (\omega - \omega_p) l_i ]$$

$$\times \prod_i \exp \left[ \frac{i}{2} k''_i(\omega_p) (\omega - \omega_p)^2 l_i \right] .$$  \hspace{1cm} (8.69)

The first term in equation (8.69) can be taken equal to 1 if $\omega_p$ is a cavity resonance frequency. The second term corresponds to the envelope group
delay, because $k_i'(\omega_p)$ is the inverse of the group velocity $v_g$. The third term corresponds to group velocity dispersion.

After linearization of the exponentials, the dispersion operator reads:

$$dT_{\text{disp}} = -\delta T_{gv} \frac{d}{dt} - iD_\omega \frac{d^2}{dt^2}, \quad (8.70)$$

with

$$\delta T_{gv} = \sum_i k_i' l_i, \quad (8.71)$$

$$D_\omega = \frac{1}{2} \sum_i k_ii' l_i. \quad (8.72)$$

The delay $\delta T_{gv}$ is the group delay induced by the intracavity optical elements. $D_\omega$ is also known as the group delay dispersion term.

### 8.4.4 Intracavity losses

The intracavity losses $\Pi$ are taken into account through the infinitesimal operator:

$$dT_{\text{losses}} = -\frac{\Pi}{2}. \quad (8.73)$$

### 8.4.5 Master equation

By gathering all the terms coming from equations (8.62), (8.66), (8.70), and (8.73), we can describe the evolution of the pulse after one round-trip through the cavity. In steady-state regime, the pulse must be unchanged after this round-trip, except for a possible time delay $\delta T$. Thus, the self-consistency of the pulse after one round-trip reads:

$$\left\{ 1 + \frac{G_0}{2} \left[ 1 - \frac{1}{\delta \omega_a} \frac{d}{dt} + \frac{1}{(\delta \omega_a)^2} \frac{d^2}{dt^2} \right] \right. \right.$$

$$\left. - \frac{\mu}{8} \omega^2 M t^2 - \delta T_M \frac{d}{dt} \right. \right.$$

$$\left. - \delta T_{gv} \frac{d}{dt} - iD_\omega \frac{d^2}{dt^2} - \frac{\Pi}{2} \right\} \mathcal{E}(t) = \mathcal{E}(t + \delta T)$$

$$= \mathcal{E}(t) + \delta T \frac{d\mathcal{E}}{dt}, \quad (8.74)$$

leading to the following so-called master equation for the field envelope:

$$\left\{ - \left( \delta T + \delta T_M + \frac{G_0}{2\delta \omega_a} \right) \frac{d}{dt} + \left( \frac{G_0}{2\delta \omega_a} - iD_\omega \right) \frac{d^2}{dt^2} - \frac{\mu}{8} \omega^2 M t^2 \right\} \mathcal{E} = \frac{\Pi - G_0}{2} \mathcal{E}. \quad (8.75)$$
8.4. TEMPORAL APPROACH II

The delay terms lead to the following mode-locking condition:

$$
\delta T = -\delta T_M - \delta T_gv - \frac{G_0}{2\delta\omega_a},
$$

(8.76)

showing that the delays due to the frequency mismatch between the modulator and the cavity, to the dispersion of the cavity and to the frequency pulling of the active medium add over one round-trip and induce a progressive time shift of the envelope of the pulse with respect to its carrier. A negative delay \(\delta T\) (which is the usual case when \(\Delta\omega = 0\)) means that the pulse envelope is late with respect to the carrier. The remaining of equation (8.75) leads to the following differential equation for the field envelope:

$$
\left(\frac{G_0}{2\delta\omega_a^2} - iD_\omega\right) \frac{d^2\mathcal{E}}{dt^2} - \frac{\mu}{8} \omega_M^2 t^2 \mathcal{E} = \frac{\Pi - G_0}{2} \mathcal{E}.
$$

(8.77)

8.4.6 Solution without dispersion \((D_\omega = 0)\)

In this case, the left-hand side of equation (8.77) leads to the same result as the model of section 8.3. Indeed, if we inject

$$
\mathcal{E}(t) = \mathcal{E}_0 \exp\left(-\frac{t^2}{2\Delta t^2}\right)
$$

(8.78)

into equation (8.77), we get:

$$
\frac{G_0}{2\delta\omega_a^2} \left(-\frac{1}{\Delta t^2} + \frac{t^2}{\Delta t^4}\right) - \frac{\mu}{8} \omega_M^2 t^2 = \frac{\Pi - G_0}{2}.
$$

(8.79)

The equation for the terms proportional to \(t^2\) leads to the same expression as equation (8.42) for the pulse duration:

$$
\Delta t = \Delta t_0 = \frac{\sqrt{2}}{\sqrt{\delta\omega_a\omega_M}} \left(\frac{G_0}{\mu}\right)^{1/4},
$$

(8.80)

and the remaining terms lead to:

$$
G_0 - \Pi = \frac{G_0}{\delta\omega_a^2 \Delta t_0^2},
$$

(8.81)

which again expresses the threshold condition because \(\delta\omega_a\Delta t \gg 1\).
8.4.7 Solution in the presence of dispersion \((D_\omega \neq 0)\)

In this case, we suppose that the dispersion term \(D_\omega\) is small and we look for the terms of smaller order in \(D_\omega\). Let us look for solutions consisting again in a Gaussian pulse, but containing a frequency chirp:

\[
E(t) = E_0 \exp\left(-\frac{t^2}{2\Delta t^2} - i\varpi t^2\right).
\]  

(8.82)

By injecting equation (8.82) into equation (8.77), we obtain:

\[
\left(\frac{G_0}{2\delta\omega_a^2} - iD_\omega\right) \left[-\frac{1}{\Delta t^2} - 2i\varpi + \left(-\frac{t}{\Delta t^2} - 2i\varpi t\right)^2\right] - \frac{\mu}{8} \frac{\omega_M^2}{\delta\omega_a^2} t^2 = \frac{\Pi - G_0}{2}.
\]

(8.83)

By solving this equation at the lowest order in \(D_\omega\), we obtain:

\[
\Delta t^4 = \frac{4G_0}{\mu\omega_M^2\delta\omega_a^2} + \frac{16D_\omega^2\delta\omega_a^2}{G_0\mu\omega_M^2} = \Delta t_0^4 + \frac{16D_\omega^2\delta\omega_a^2}{G_0\mu\omega_M^2},
\]

(8.84)

\[
\varpi = \frac{D_\omega}{2} \sqrt{\frac{\mu\omega_M^2\delta\omega_a^6}{G_0^2}}.
\]

(8.85)

We can thus see that the intracavity dispersion plays two roles: it increases the pulse duration and it creates a frequency chirp in the pulse. Indeed, the instantaneous frequency of the pulse of equation (8.82) is given by the derivative of its phase:

\[
\omega(t) = \omega_p + 2\varpi t.
\]

(8.86)
This shows that the instantaneous frequency of light varies linearly during the pulse. Depending on the sign of the dispersion, the blue (respectively red) part of the spectrum is at the beginning (respectively at the end) or at the end (respectively at the beginning) of the pulse. In the case of positive dispersion \((D_\omega > 0)\), which is the most common case in laser amplifiers, the chirp is positive, as illustrated in figure 8.9: the red part of the spectrum propagates at the front of the pulse.

### 8.5 Temporal approach to passive mode-locking

In the case of passive mode-locking, the optical modulation is not created by a modulator driven by an external modulation, but by an intracavity nonlinear effect that is driven by the light pulse itself. In order to favor a pulsed regime, this nonlinearity must be quasi-instantaneous in order to follow the variations of the intensity of the pulse and must lead to a decrease of the losses when the intensity increases. Among these nonlinearities, we can mention the fast saturable absorbers and the Kerr lenslike effects.

In the case of the fast saturable absorber, a thin absorbing medium is coated on one of the mirrors or introduced inside the cavity. As long as the laser intensity is low, this absorber introduces constant losses. When the intensity reaches a value comparable with the saturation intensity of the absorber, the absorption decreases and the losses are reduced. This favors the high peak intensities associated with short pulses.

The same kind of effect can occur when one introduces a Kerr medium inside the cavity. In this case, when this Kerr medium is shined by an intense Gaussian beam, it behaves like a positive lens (the refractive index increases with the intensity). Under some conditions, this can lead to a reduction of the radius of the laser mode on one of the mirrors (or on any other intracavity aperture) and consequently to a decrease of the diffraction losses. Here again, the losses of the laser are decreased in the presence of a large intracavity intensity, thus favoring the pulsed behavior. This effect is known as the Kerr lens mode-locking. Of course, this Kerr effect also introduces self-phase-modulation that must be taken into account.

To describe the behavior of such a passively mode-locked laser, one also usually uses the formalism derived in the preceding section, taking the new ingredients into account.
8.5.1 Fast saturable absorber

A fast saturable absorber behaves like a two-level system with a relaxation time much shorter than the pulse duration. This can occur in some semiconductor materials. In these conditions, the saturation is instantaneous and the absorption coefficient reads:

$$\alpha_{\text{abs}}(t) = \alpha_{0,\text{abs}} \left(1 + \frac{I(t)}{I_{\text{sat,abs}}} \right)^{-1}. \quad (8.87)$$

Since, according to equation (1.6), $I(t)$ is proportional to $|\mathcal{E}(t)|^2$, we have, in the case of a weak saturation and for an absorber of length $L_{\text{abs}}$:

$$d T_{\text{abs}} \simeq -\frac{\alpha_{0,\text{abs}}}{2} L_{\text{abs}} + \frac{\alpha_{0,\text{abs}}}{2} L_{\text{abs}} \frac{2\varepsilon_0 n_0 c_0 |\mathcal{E}|^2}{I_{\text{sat,abs}}}. \quad (8.88)$$

The first term can be incorporated in the losses $\Pi$, leaving us with:

$$d T_{\text{abs}} = \varepsilon_{\text{abs}} |\mathcal{E}|^2. \quad (8.89)$$

8.5.2 Kerr effect

In a medium exhibiting Kerr effect, the refractive index depends on the instantaneous intensity:

$$n = n_0 + n_2 I. \quad (8.90)$$

This creates a phase shift which depends on the intensity, leading to:

$$d T_{\text{Kerr}} = i\delta_K |\mathcal{E}|^2. \quad (8.91)$$

8.5.3 Master equation

In the case of a passively mode-locked laser, the modulation of the losses is always in phase with the pulse. Consequently, all the delay terms compensate and the term proportional to $d/dt$ in the master equation disappears. The differential equation (8.77) is thus replaced by:

$$\left( \frac{G_0}{2\delta_\omega^2} - iD_\omega \right) \frac{d^2 \mathcal{E}}{dt^2} + (\varepsilon_{\text{abs}} + i\delta_K) |\mathcal{E}|^2 \mathcal{E} = \frac{\Pi - G_0}{2} \mathcal{E}. \quad (8.92)$$

The resolution of this equation is beyond the scope of the present lecture. We will only mention that under well chosen circumstances, it can lead to
8.5. PASSIVE MODE-LOCKING

Figure 8.10: (a) Intensity and (b) instantaneous frequency of a soliton-like pulse generated by a passively mode-locked laser.

the so-called soliton-like mode-locking in which the envelope of the pulse of the pulse is given by the following expression:

\[ E(t) = \mathcal{E}_0 \left[ \cosh \left( \frac{t}{\Delta t} \right) \right]^{1+iy} . \]  

(8.93)

Equation (8.93) leads to the following intensity:

\[ I(t) = \frac{I_0}{\cosh^2 \left( \frac{t}{\Delta t} \right)} , \]  

(8.94)

and to the following instantaneous frequency:

\[ \omega(t) = \omega_0 + \frac{y}{\Delta t} \tanh \frac{t}{\Delta t} . \]  

(8.95)

The intensity and the instantaneous frequency of such a pulse are reproduced in Figure 8.10.
8. MODE-LOCKING
Chapter 9

Propagation and characterization of short laser pulses

In this chapter, we study how the short pulses generated by the mode-locked lasers described in chapter 8 propagate in optical media. In particular, we will show in section 9.1 that the dispersion of materials is responsible for the propagation velocity of the pulse envelope, for its broadening, and for the fact that it acquires a chirp. We will also calculate how a chirped pulse can be compressed by propagation through a dispersive medium. Section 9.2 presents a quick overview of some experimental methods aiming at measuring the characteristics of short pulses.

9.1 Dispersion. Chirp

9.1.1 Time-frequency Fourier transform: some useful definitions and relations

Let us summarize here the notations that we use for Fourier transformations and some well know relations about the Fourier transform of Gaussians. Let us consider a function of time \( f(t) \). Its Fourier transform (if it exists) is the function \( \hat{f}(\omega) \) defined by:

\[
\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt .
\]  

(9.1)

The reverse transformation reads:

\[
f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{-i\omega t} d\omega .
\]  

(9.2)
In the case of a Gaussian:
\[ f(t) = \exp \left( -\frac{t^2}{2\Delta t_0^2} \right), \]  
(9.3)
The Fourier transform is also a Gaussian:
\[ \tilde{f}(\omega) = \Delta t \exp \left( -\frac{\omega^2}{2\Delta \omega^2} \right), \]  
(9.4)
with
\[ \Delta \omega \Delta t_0 = 1. \]  
(9.5)
We also mention the following useful relation which is valid even for complex values of \( \delta \):
\[ \int_{-\infty}^{\infty} \exp \left[ -\frac{(u + i\alpha)^2}{\delta^2} \right] du = \sqrt{\pi} \delta. \]  
(9.6)

9.1.2 Propagation of a Gaussian pulse in a dispersive medium

In this subsection, we calculate the propagation of a Gaussian pulse of light in a dispersive optical medium.

9.1.2.1 Description of the pulse at the input of the medium

Let us consider a Gaussian pulse of light propagating along \( z \) and launched in \( z = 0 \) at the input of a dispersive medium:
\[ E(0, t) = \mathcal{E}(0, t)e^{-i\omega_0 t} + \text{c.c.}, \]  
(9.7)
with the following Gaussian envelope:
\[ \mathcal{E}(0, t) = E_0 \exp \left( -\frac{t^2}{2\Delta t_0^2} \right). \]  
(9.8)
Then the field of equation (9.7) can be re-written as:
\[ E(0, t) = E^{(+)}(0, t) + \text{c.c.}, \]  
(9.9)
with the following expression for the positive frequency part of the field (also known as the analytical signal):
\[ E^{(+)}(0, t) = E_0 \exp \left( -\frac{t^2}{2\Delta t_0^2} \right)e^{-i\omega_0 t}. \]  
(9.10)
9.1. DISPERSION. CHIRP

The Fourier transform of this analytical signal [which is also the positive frequency part of the Fourier transform of the whole electric field of equation (9.9)] is hence given by:

\[ \tilde{E}^{(+)}(0, \omega) = \tilde{E}_0 \exp \left( -\frac{(\omega - \omega_p)^2}{2\Delta \omega^2} \right), \]  

with

\[ \tilde{E}_0 = \Delta t_0 E_0. \]  

Then the field at the input of the dispersive medium can be seen as the sum of monochromatic waves:

\[ E^{(+)}(0, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega \, e^{-i\omega t} \tilde{E}^{(+)}(0, \omega) , \]  

which all propagate along \( z \) with their own wavenumber \( k(\omega) \), accumulating a spectral phase \( k(\omega)z \), leading to:

\[ E^{(+)}(z, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega \, \tilde{E}^{(+)}(0, \omega) \, e^{-i\omega t} \, e^{ik(\omega)z} . \]  

9.1.2.2 Dispersion of the medium

The wavenumber \( k(\omega) \) is given by:

\[ k(\omega) = n_0(\omega) \frac{\omega}{c_0} . \]  

We perform a Taylor expansion at second order\(^1\) around the central pulse frequency \( \omega_p \):

\[ k(\omega) \simeq k(\omega_p) + (\omega - \omega_p) \frac{dk}{d\omega} \bigg|_{\omega_p} + \frac{1}{2} (\omega - \omega_p)^2 \frac{d^2 k}{d\omega^2} \bigg|_{\omega_p} . \]  

The second term of this expansion contains the group velocity defined by:

\[ v_g = \left( \frac{dk}{d\omega} \right)^{-1} = \frac{c_0}{n_0 + \omega \frac{dn_0}{d\omega}} = \frac{c_0}{n_0 - \lambda \frac{dn_0}{d\lambda}} . \]  

The third term is known as the second-order dispersion or group delay dispersion:

\[ \beta_2 = \frac{d^2 k}{d\omega^2} \bigg|_{\omega_p} . \]
Table 9.1: Some examples of material dispersion parameters for two different glasses. $T_g$ is the group-delay for a 1 mm thickness.

<table>
<thead>
<tr>
<th>Material</th>
<th>$\lambda_0$ (nm)</th>
<th>$n_0$</th>
<th>$T_g$ (fs/mm)</th>
<th>$\beta_2$ (fs$^2$/mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>BK7</td>
<td>400</td>
<td>1.5308</td>
<td>5282</td>
<td>120.79</td>
</tr>
<tr>
<td></td>
<td>500</td>
<td>1.5214</td>
<td>5185</td>
<td>86.87</td>
</tr>
<tr>
<td></td>
<td>600</td>
<td>1.5163</td>
<td>5136</td>
<td>67.52</td>
</tr>
<tr>
<td></td>
<td>800</td>
<td>1.5108</td>
<td>5092</td>
<td>43.96</td>
</tr>
<tr>
<td></td>
<td>1000</td>
<td>1.5075</td>
<td>5075</td>
<td>26.93</td>
</tr>
<tr>
<td></td>
<td>1200</td>
<td>1.5049</td>
<td>5069</td>
<td>10.43</td>
</tr>
<tr>
<td>SF10</td>
<td>400</td>
<td>1.7783</td>
<td>6626</td>
<td>673.68</td>
</tr>
<tr>
<td></td>
<td>500</td>
<td>1.7432</td>
<td>6163</td>
<td>344.19</td>
</tr>
<tr>
<td></td>
<td>600</td>
<td>1.7267</td>
<td>5980</td>
<td>233.91</td>
</tr>
<tr>
<td></td>
<td>800</td>
<td>1.7112</td>
<td>5830</td>
<td>143.38</td>
</tr>
<tr>
<td></td>
<td>1000</td>
<td>1.7038</td>
<td>5771</td>
<td>99.42</td>
</tr>
<tr>
<td></td>
<td>1200</td>
<td>1.6992</td>
<td>5743</td>
<td>68.59</td>
</tr>
</tbody>
</table>

Some examples of values of these parameters are given in table 9.1. It should be noticed that since transparent optical materials are used in the so-called “normal” dispersion region, one has $\beta_2 > 0$.

### 9.1.2.3 Pulse propagation and broadening

Using equations (9.14, 9.16-9.18), the pulse at abscissa $z$ reads:

$$E^{(+)}(z, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega \tilde{E}_0 \exp \left[ -\frac{(\omega - \omega_p)^2}{2\Delta \omega^2} \right]$$

$$\times \exp \left[ -i\omega t + ik_p z + i(\omega - \omega_p) \frac{z}{v_g} + i(\omega - \omega_p)^2 \frac{\beta_2 z^2}{2} \right]$$

(9.19)

where we have introduced

$$k_p = k(\omega_p).$$

(9.20)

We introduce the change of variable

$$u = \omega - \omega_p,$$

(9.21)

---

1This expansion is valid only for relatively long pulses propagating in slightly dispersive media. Of course, for ultrashort pulses or for strongly dispersive media, one must take higher order terms into account, or even avoid this kind of Taylor expansion.
9.1. DISPERSION. CHIRP

leading to:

$$E^{(+)}(z,t) = \frac{\bar{E}_0}{\sqrt{2\pi}} e^{-i(\omega_p t - k_p z)} \int_{-\infty}^{\infty} du \exp \left[ -\frac{u^2}{2\Delta u^2} - iu \left( t - \frac{z}{v_g} \right) \right],$$  \hspace{1cm} (9.22)

with

$$\frac{1}{\Delta u^2} = \frac{1}{\Delta \omega^2} - i\beta_2 z.$$  \hspace{1cm} (9.23)

Using equation (9.6), we eventually obtain:

$$E^{(+)}(z,t) = \Delta u \Delta t_0 \bar{E}_0 e^{-i(\omega_p t - k_p z)} \exp \left[ -\frac{(t - \frac{z}{v_g})^2}{2\Delta t(z)^2} \right] \exp \left[ -i\frac{(t - \frac{z}{v_g})^2}{2\Delta t(z)^2} \beta_2 \Delta \omega^2 z \right],$$  \hspace{1cm} (9.24)

with

$$\Delta t(z)^2 = \Delta t_0^2 + \Delta \omega^2 \beta_2 z^2.$$  \hspace{1cm} (9.25)

Equation (9.24) shows that the envelope of the pulse travels along $z$ at the group velocity $v_g$. The two last terms of this equation show that the dispersion $\beta_2 \neq 0$ of the material is responsible for the pulse broadening described in equation (9.25) and for the apparition of a chirp. Since, as shown for example in table 9.1, transparent media usually have a positive group velocity dispersion, the pulses acquire a positive chirp (see figure 8.9) when they propagate through ordinary optical media. Figure 9.1 reproduces the evolution of the pulse duration $\Delta t(z)$ given by equation (9.25) for $\beta_2 = 100 \text{ fs}^2/\text{mm}$ and

![Figure 9.1: Evolution of the duration of a pulse propagating through a dispersive medium with $\beta_2 = 100 \text{ fs}^2/\text{mm}$ for two initial pulse durations $\Delta t_0 = 10 \text{ fs}$ and $\Delta t_0 = 100 \text{ fs}$.](image-url)
for two values of the initial pulse durations: \( \Delta t_0 = 10 \) fs and \( \Delta t_0 = 100 \) fs. The role of the pulse spectral width \( \Delta \omega = 1/\Delta t_0 \) is particularly striking.

### 9.1.3 Propagation of a chirped pulse in a dispersive medium

Let us now consider the propagation of an initially chirped pulse in a dispersive medium. The positive frequency part of the electric field of the incident pulse is supposed to be given by:

\[
E^{(+)}(0,t) = E_0 \exp \left[ -\frac{1 + iC}{2} \frac{t^2}{\Delta t_0^2} \right] e^{-i\omega_p t},
\]

where \( C \) is the chirp rate. The Fourier transform of \( E^{(+)}(0,t) \) is:

\[
\tilde{E}^{(+)}(0,\omega) = \tilde{E}_0 \exp \left[ -\frac{(\omega - \omega_p)^2}{2(1 + iC)} \Delta t_0^2 \right]
= \tilde{E}_0 \exp \left[ -\frac{\Delta t_0^2 (1 - iC)}{2(1 + C^2)} (\omega - \omega_p)^2 \right],
\]

showing that the spectral width of the pulse is \( \sqrt{1 + C^2}/\Delta t_0 \) instead of \( 1/\Delta t_0 \) for a non chirped pulse.

After propagation through a length \( z \) of a dispersive medium characterized by a group velocity dispersion coefficient \( \beta_2 \), the pulse spectrum becomes:

\[
\tilde{E}^{(+)}(z,\omega) \propto \tilde{E}^{(+)}(0,\omega) \exp \left[ i\frac{\beta_2}{2} (\omega - \omega_p)^2 z \right]
\propto \tilde{E}_0 \exp \left[ -\frac{(\omega - \omega_p)^2}{2(1 + C)} \left( \frac{\Delta t_0^2}{2(1 + C^2)} - \frac{i}{2} \beta_2 z \right) \right].
\]

Consequently, the pulse shape becomes:

\[
E^{(+)}(z,t) \propto E_0 \exp \left[ -\frac{(t - z/v_g)^2}{4} \left( \frac{\Delta t_0^2}{2(1 + C)} - \frac{i}{2} \beta_2 z \right)^{-1} \right]
\propto E_0 \exp \left[ -\frac{(t - z/v_g)^2}{2} \frac{\Delta t_0^2 + i(C \Delta t_0^2 + \beta_2 C^2 z + \beta_2 z)}{(\Delta t_0^2 + \beta_2 C^2 z)^2 + \beta_2^2 z^2} \right].
\]

From equation (9.29), the pulse duration is given by:

\[
\Delta t(z) = \Delta t_0 \sqrt{\left( 1 + \frac{\beta_2 C z}{\Delta t_0^2} \right)^{2} + \left( \frac{\beta_2 z}{\Delta t_0^2} \right)^{2}}.
\]
Figure 9.2 represents the evolution of $\Delta t(z)$ versus $z$ in two different cases.

\[ \Delta t_0 = 50 \text{ fs} \]
\[ C = 3 \]
\[ \beta_2 = 100 \text{ fs}^2/\text{mm} \]
\[ \beta_2 = -100 \text{ fs}^2/\text{mm} \]

Figure 9.2: *Evolution of the duration of a chirped pulse propagating through a dispersive medium with $\beta_2 = \pm 100 \text{ fs}^2/\text{mm}$ for an initial pulse duration $\Delta t_0 = 50 \text{ fs}$ and a chirp parameter $C = 3$.*

One can see that the pulse can be compressed provided $C\beta_2 < 0$. In this case, $\Delta t(z)$ reaches a minimum at abscissa $z_{\text{min}}$ given by:

\[ z_{\text{min}} = -\frac{C}{1 + C^2} \frac{\Delta t_0^2}{\beta_2}. \]  

The corresponding minimum pulse duration is given by:

\[ \Delta t_{\text{min}} = \frac{\Delta t_0}{\sqrt{1 + C^2}}, \]  

which is consistent with the fact that the spectral width of the pulse is $\sqrt{1 + C^2}/\Delta t_0$.

Figure 9.3: *Adjustable group delay dispersion created by four prisms.*

It is thus clear that to compress a pulse exhibiting a positive chirp, one needs a negative dispersion. This cannot be achieved by propagation in a
bulk transparent material, but requires more sophisticated techniques based on the use of a negatively dispersive fiber, a Gires-Tournois interferometer, a pair of prisms, chirped mirrors, a pair of diffraction gratings, or more complicated arrangements. Figure 9.3 shows for example how four prisms are able to create an adjustable group delay dispersion. Indeed, in this system, the angular dispersion of the prisms allows to create a longer path for the red spectral components than for the blue spectral components, i. e., a negative group delay dispersion. This negative angular group delay dispersion can be partially compensated for by translating the last prism along the arrow, thus introducing more or less glass in the beam propagation.

9.2 Pulse characterization

Since optical detectors are quadratic, it is of course impossible to directly measure the electric field of a light pulse. Concerning ultra-short pulses, it is even impossible to directly measure the evolution of the intensity versus time during the pulse because the fastest detectors exhibit bandwidths limited to a few hundreds of GHz. Different strategies based on optical correlation techniques have been developed to characterize ultrashort pulses. It is of course beyond the scope of the present lecture to give a precise overview of these techniques. In the following, we just attempt to explain the operating principles of a few of them.

9.2.1 Intensity autocorrelation

![Non-collinear intensity autocorrelator](image)

Figure 9.4: Non-collinear intensity autocorrelator. BS: beamsplitter; M: mirror.
9.2. PULSE CHARACTERIZATION

The principle of the intensity autocorrelator is schematized in figure 9.4. The pulse to be analyzed is split into two pulses by a beam-splitter. An adjustable delay \( \tau \) is applied to one of the pulses with respect to the other one before they get recombined inside a nonlinear medium, for example here a nonlinear crystal providing second-harmonic generation. Notice that in the case of the scheme of figure 9.4, the two beams are recombined at a mutual angle in the nonlinear crystal. This allows the second harmonic intensity emitted along the bisector direction of the two incident beams to be proportional to the product of the intensities of the two pulses:

\[
I_{\text{SHG}}(t, \tau) \propto I(t)I(t + \tau) \propto |E(t)|^2 |E(t + \tau)|^2 .
\] (9.33)

However, since the detector is too slow to follow the fast variations of equation (9.33), it provides a signal proportional to the intensity autocorrelation given by equation (9.34) on a dark background:

\[
S_{\text{intens,ac}}(\tau) = \int_{-\infty}^{\infty} |E(t)|^2 |E(t + \tau)|^2 \, dt .
\] (9.34)

We recall that the electric field of the pulse is defined as:

\[
E(t) = E^{(+)}(t) + \text{c.c.} = \mathcal{E}(t)e^{-i\omega_p t} + \text{c.c.} = |\mathcal{E}(t)|e^{i\varphi(t)}e^{-i\omega_p t} + \text{c.c.} .
\] (9.35)

It is worth noticing that the intensity autocorrelation is symmetrical:

\[
S_{\text{intens,ac}}(-\tau) = S_{\text{intens,ac}}(\tau) .
\] (9.36)

Some examples of intensity autocorrelation traces are shown in figure 9.5 for three different pulse shapes. Note that the autocorrelation loses details of the pulse, and, as a result, all of these pulses have similar autocorrelations. It is clear from this example that it is impossible to guess the shape of the pulse from the shape of the autocorrelation. It is even impossible to deduce the pulse duration from the autocorrelation without assuming a given shape for the pulse.

### 9.2.2 Interferometric autocorrelation

According to equation (9.34), the intensity autocorrelation carries absolutely no information on the evolution of the phase \( \varphi(t) \) of the field during the pulse.
For example, one can absolutely not guess whether a pulse carries a chirp from the shape of its intensity autocorrelation. To (partially) circumvent this drawback, the so-called interferometric interferometer, schematized in figure 9.6, is sometimes used. In this autocorrelator, the two time shifted copies of the pulse are sent collinearly on the SHG crystal. The SHG power is thus proportional to the square of the interference pattern between the two pulses. Consequently, the signal provided by the detector is proportional to:

$$S_{\text{interf,ac}}(\tau) = \int_{-\infty}^{\infty} \left\{ |E^{(+)}(t) + E^{(+)}(t + \tau)|^2 \right\}^2 \, dt ,$$  \hspace{1cm} (9.37)

Using again equation (9.35), we can identify in equation (9.37) the terms which do not depend on $\tau$, those who are proportional to $\exp(\pm i\omega_p \tau)$, and
9.2. PULSE CHARACTERIZATION

Figure 9.6: Interferometric autocorrelator. BS: beamsplitter; M: mirror.

those who are proportional to exp (±2iωpτ), leading to:

\[
S_{\text{interf,ac}}(\tau) = 12 \int_{-\infty}^{\infty} \left[ |E(t)|^4 + 2 |E(t)|^2 |E(t + \tau)|^2 \right] dt \\
+ 12 e^{i\omega_p\tau} \int_{-\infty}^{\infty} |E(t)||E(t + \tau)||[|E(t)|^2 + |E(t + \tau)|^2] e^{[\phi(t) - \phi(t + \tau)]} dt + \text{c.c.} \\
+ 6 e^{2i\omega_p\tau} \int_{-\infty}^{\infty} |E(t)|^2 |E(t + \tau)|^2 e^{2[\phi(t) - \phi(t + \tau)]} dt + \text{c.c.} .
\] (9.38)

Two examples of interferometric autocorrelations are reproduced in figure 9.7. A characteristic feature of interferometric autocorrelations is the 8:1 ratio between the peak and the wings. This can be seen from equation (9.37):

\[
\frac{S_{\text{interf,ac}}(0)}{S_{\text{interf,ac}}(\infty)} = \frac{\int_{-\infty}^{\infty} |E^+(t)|^4 dt + \int_{-\infty}^{\infty} |E^+(t)|^4 dt}{\int_{-\infty}^{\infty} |E^+(t)|^4 dt + \int_{-\infty}^{\infty} |E^+(t)|^4 dt} = \frac{16 \int_{-\infty}^{\infty} |E^+(t)|^4 dt}{2 \int_{-\infty}^{\infty} |E^+(t)|^4 dt} = 8 .
\] (9.39)

Figure 9.7 shows that the interferometric autocorrelation is somewhat sensitive to the phase \(\phi(t)\) of the pulse. More insight can be gained inside the evolution of this phase by Fourier filtering the signal in order to separate the three components of equation (9.38). However, even with these refinements, this technique can only provide partial knowledge on the intensity and the phase of the pulse. It cannot fully characterize arbitrarily shaped pulses.
9. SHORT PULSES

Figure 9.7: Two ultrashort pulses (a) and (b) with their respective interferometric autocorrelations (c) and (d). The two pulses have the same intensities but the pulse in (b) has a positive chirp which is absent in the pulse in (a). This leads to a visible difference between the two corresponding autocorrelations. In (d), the chirp washes out the wings of the autocorrelation. Note the ratio 8:1 (peak to the wings), characteristic of interferometric autocorrelation traces. (Taken from Wikipedia).

9.2.3 Frequency-Resolved Optical Gating (FROG)

As we have just seen, pulse autocorrelations do not allow to determine the intensity and the phase of arbitrarily shaped pulses. To do so, one needs not only to perform autocorrelation measurements in the time domain, but more generally in the joint time-frequency domain. This means that the autocorrelation must be measured both versus time and versus frequency to allow a complete determination of the pulse shape.

The idea behind such techniques is the same one as the usual representation of music: we are looking for a method to represent the evolution of the spectrum of the pulse versus time. This type of representation is called a spectrogram. A spectrogram of the analytic signal $E^{(+)}(t)$ is given by:

$$S_{\text{spectrogr}}(\omega, \tau) = \left. \left( \int_{-\infty}^{\infty} E^{(+)}(t) g(t - \tau) e^{i\omega t} dt \right)^2 \right|,$$

(9.40)

where $g(t - \tau)$ is a gating function, which plays the role of a time window. The meaning of equation (9.40) is clear: the spectrogram at time $\tau$ is the
power spectral density of a small portion of the field around time $\tau$, as shown in figure 9.8. Thus, a typical spectrogram can be interpreted as the time evolution of a “local” spectrum of the laser.

The question which arises now is “how do we build a gating function $g(t-\tau)$ for an ultrashort light pulse?” The idea is to use the nonlinear effect created by a delayed version of the pulse itself in a well chosen nonlinear medium. There are many ways to do this. We mention here only the so-called polarization-gate FROG, which is the one which is conceptually the closest to the spectrogram of equation (9.40). Its principle is described in figure 9.9. It uses the electronic Kerr effect (a third order nonlinearity) in a nonlinear medium, such as fused silica. More precisely, the pulse creates a birefringence proportional to its intensity in the medium. Then, in a crossed polarizer arrangement, one detects this birefringence on the other pulse. The field which crosses the second polarizer is consequently proportional to the
probe field $E^{(+)}(t)$ and to the birefringence induced by the time delayed field which is proportional to its intensity $|E^{(+)}(t-\tau)|^2$. This latter intensity constitutes the gate function $g(t-\tau)$ leading to:

$$S_{\text{spectro}}(\omega, \tau) \propto \left| \int_{-\infty}^{\infty} E^{(+)}(t) E^{(+)}(t-\tau) e^{i\omega t} dt \right|^2,$$

(9.41)

the Fourier transform being provided by the spectrometer and the camera.

Some examples of spectrograms obtained by this technique are shown in figure 9.10. They are taken from papers written by Rick Trebino. It is clear
that such figures provide an intuitive view of the intensity and frequency content of the pulse. Moreover, there exist algorithms that permit to unambiguously retrieve the pulse phase and intensity from the spectrogram. This problem, together with other types of FROG, other time-frequency pulse measurement methods, or competing methods such as the “spectral phase interferometry for direct electric-field reconstruction” (SPIDER), lie beyond the scope of the present lecture.
9. SHORT PULSES
Chapter 10
Optical resonators I: Modes and rays

10.1 Introduction: the concept of mode

Up to now, we have considered that the field distribution inside the cavity was that of a truncated plane wave. This is of course wrong and the aim of this chapter and of the following is to determine the exact field distribution(s) in the cavity, taking its geometric characteristics into account.

10.1.1 Propagation kernel

The monochromatic intracavity field mainly propagates along the cavity axis $z$ (see figure 10.1). We can thus suppose that it is the product of a plane wave by a slowly varying complex envelope $U(x, y, z)$, namely:

$$E(x, y, z, t) = U(x, y, z)e^{-i(ωt−kz)} + c.c. .$$

(10.1)

During its propagation inside the cavity, the field undergoes reflections on the mirrors, diffraction by intracavity apertures, lens effects, etc. This can be described by the following general linear transformation which relates the field amplitude $U^{(1)}(x, y, z_0)$ after one round-trip in a given plane $z = z_0$ to the field in the same plane one round-trip earlier $U^{(0)}(x, y, z_0)$:

$$U^{(1)}(x, y, z_0) = \int \int K(x, y, x_0, y_0, z_0) U^{(0)}(x_0, y_0, z_0) dx_0 dy_0 .$$

(10.2)

The propagation kernel $K(x, y, x_0, y_0)$ appearing in this equation contains the details of the cavity. It depends of course on the plane $z_0$ in which the mode is considered.
10.1.2 Eigenmode

An eigenmode of this cavity in the plane $z_0$ is an eigenvector of the propagation equation (10.2), thus verifying:

$$\gamma_{mn}U_{mn}(x, y, z_0) = \int \int K(x, y, x_0, y_0) U_{mn}(x_0, y_0, z_0) \, dx_0 \, dy_0,$$  \hspace{1cm} (10.3)

where $\gamma_{mn}$ is the associated (complex) eigenvalue. The two indices $m$ and $n$ label the considered transverse mode along two transverse dimensions. Their possible meanings will be discussed in chapter 11. In this case, equation (10.2) becomes:

$$U_{mn}^{(1)}(x, y, z_0) = \gamma_{mn} U_{mn}^{(0)}(x_0, y_0, z_0).$$ \hspace{1cm} (10.4)

Since the cavity of figure 10.1 contains no gain, we have $|\gamma_{mn}| < 1$ and the diffraction losses per cavity round-trip of the considered eigenmode are

$$\Pi_{\text{dfr}} = 1 - |\gamma_{mn}|^2.$$ \hspace{1cm} (10.5)

The mode with the lowest losses, if it exists, is usually called the ‘fundamental mode’ of the cavity.

If we want the mode of equation (10.3) to be resonant inside the cavity, its frequency needs to satisfy the following relation:

$$\arg(\gamma_{mn}) + kL_{\text{cav,opt}} = 2p\pi,$$ \hspace{1cm} (10.6)
10.1. CONCEPT OF MODE

where $p$ is an integer. This leads to the following eigenfrequency for that particular mode:

$$\omega_{mn} = 2\pi \frac{c_0}{L_{cav,opt}} \left( p - \frac{\arg(\gamma_{mn})}{2\pi} \right). \quad (10.7)$$

One should notice that since the transformation of equation (10.2) is in general non Hermitian, nothing guarantees that at least one eigenmode exists. However, up to now, it seems that all lasers have succeeded in determining their transverse mode(s) structure...

10.1.3 The Fox and Li approach

Historically, the first successful approach to find the fundamental mode of a cavity was performed numerically by Fox and Li in 1961. They numerically sent an arbitrary flat wavefront into a two-plane-mirror “strip resonator”, i.e., a linear resonator with two mirrors exhibiting an infinite length along one, say $y$, transverse dimension. They used then the Huygens-Fresnel integral to compute the evolution of the intracavity field along its successive round-trips inside the cavity, as reproduced in figure 10.2. They observed that after a

![Figure 10.2](image)

Figure 10.2: Typical results of Fox and Li’s numerical calculations. The uniform distribution shown in (a) is launched inside the cavity. (b) Field pattern after one round-trip inside the cavity. (c) Steady-state field distribution after 300 round-trips (Taken from A. E. Siegman, Lasers, op. cit.).
large number of round-trips, the field profile stabilizes to the lowest-order eigenmode, as shown in figure 10.2(c). In particular, one can see from figure 10.2(c) that the field does not vanish at the extremity of the mirror. The ratio of the field after one round-trip permits to determine the losses of this eigenmode.

10.1.4 An analytical approach?

We will see in the following that the eigenmodes of so-called stable cavities can be determined analytically. Before this, we need to define a certain number of tools, among which ray matrices play a particularly important role.

10.2 Ray optics and ray matrices

It is a particularly striking phenomenon that ray matrices, also known as $ABCD$ matrices, which were developed and taught long ago to describe the propagation of rays in the approximation of geometrical optics, are now commonly used to describe the propagation of laser beams. We will thus in this section review the basic laws of geometrical optics using these matrices.

10.2.1 Vector description of optical rays

In this section, we describe light as paraxial rays, i.e., rays of light which are “close enough” to a given propagation direction called $z$. This means that, in two dimensions, these rays can be described, in a given plane perpendicular to $z$, by their distance $r(z)$ from the $z$ axis and their slope $dr/dz$ with respect to this axis. Such a ray is reproduced in figure 10.3. More generally, in the case of propagation in a medium of refractive index $n_0(z)$ in plane $z$, we define the “reduced slope” $r'(z)$ by:

$$r'(z) \equiv n_0(z) \frac{dr(z)}{dz}.$$  \hspace{1cm} (10.8)

The ray can then be described by the vector $r(z)$:

$$r(z) = \begin{pmatrix} r(z) \\ r'(z) \end{pmatrix}.$$  \hspace{1cm} (10.9)
10.2.2 Ray matrices

10.2.2.1 Free space

Let us consider the propagation of a ray from \( z = z_1 \) to \( z = z_2 = z_1 + L \), as shown in figure 10.4. Then we have:

\[
\begin{align*}
 r'_2 &= r'_1 , \\
 r_2 &= r_1 + \frac{dr}{dz} \bigg|_{z_1} L = r_1 + r'_1 \frac{L}{n_0} ,
\end{align*}
\]

(10.10) \hspace{1cm} (10.11)

where \( n_0 \) is the refractive index of the medium in which the propagation takes place. Equations (10.10) and (10.11) can be summarized as:

\[
\mathbf{r}_2 = \begin{pmatrix} r_2 \\ r'_2 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} r_1 \\ r'_1 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mathbf{r}_1 ,
\]

(10.12)

where the \( ABCD \) matrix for propagation through a length \( L \) of material of refractive index \( n_0 \) is:

Length \( L \):

\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & L/n_0 \\ 0 & 1 \end{pmatrix} .
\]

(10.13)
10.2.2.2 Thin lens

One can similarly determine the \(ABCD\) matrix for a thin lens of focal length \(f\):

\[
\text{Thin lens: } \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix},
\]  \hspace{1cm} (10.14)

with the convention \(f > 0\) for a positive lens.

10.2.2.3 Curved mirror

In the case of a ray reflected by a curved mirror (see figure 10.5) of radius of curvature \(R\) (with \(R > 0\) for a concave mirror), one must distinguish between normal incidence \((\theta = 0)\) and arbitrary incidence \((\theta \neq 0)\).

\[\text{Figure 10.5: Reflection on a curved mirror with angle of incidence } \theta.\]

**Curved mirror at normal incidence \((\theta = 0)\).** A curved mirror of radius of curvature \(R\) is then equivalent to a thin lens of focal length \(R/2\). Then, according to equation (10.14), its \(ABCD\) matrix is:

\[
\text{Mirror \((\theta = 0)\): } \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2/R & 1 \end{pmatrix},
\]  \hspace{1cm} (10.15)

**Curved mirror at arbitrary incidence \((\theta \neq 0)\).** In this case the mirror behaves differently in the tangential plane (incidence plane) and in the sagittal plane. It is equivalent to a curved mirror at normal incidence with an effective radius of curvature \(R_e\) which is different in the two planes:

\[\begin{align*}
R_e &= R \cos \theta \text{ in the tangential plane }, \\
R_e &= R/ \cos \theta \text{ in the sagittal plane },
\end{align*}\]  \hspace{1cm} (10.16)  \hspace{1cm} (10.17)
leading to:

\[
\text{Mirror } (\theta \neq 0): \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2/R_e & 1 \end{pmatrix}.
\]

\[ (10.18) \]

### 10.2.2.4 Curved dielectric interface

Let us consider a curved interface of radius of curvature \( R \) \((R > 0 \text{ for a concave interface})\) separating two materials of refractive indices \( n_1 \) and \( n_2 \) (see figure 10.6). Here again, we must distinguish between normal and oblique incidence.

**Curved interface at normal incidence** \((\theta_1 = 0)\). This case is represented in figure 10.6. The \(ABCD\) matrix is given by:

\[\text{Curved interface } (\theta_1 = 0): \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ (n_2 - n_1)/R & 1 \end{pmatrix}.\]

\[ (10.19) \]

**Curved interface at arbitrary incidence** \((\theta_1 \neq 0)\). The situation is a bit more complicated here, obliging us to treat separately the tangential and sagittal planes.

**Curved interface at arbitrary incidence** \((\theta_1 \neq 0), \text{ tangential plane} \). This case is represented in figure 10.7. The \(ABCD\) matrix is given by:

\[
\text{Curved interface } (\theta_1 \neq 0), \text{ tangential plane:} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \cos \theta_2/\cos \theta_1 & 0 \\ \Delta n_e/R & \cos \theta_1/\cos \theta_2 \end{pmatrix},
\]

\[ (10.20) \]

with

\[
\Delta n_e = (n_2 \cos \theta_2 - n_1 \cos \theta_1)/\cos \theta_2 \cos \theta_1.
\]

\[ (10.21) \]
Curved interface at arbitrary incidence ($\theta_1 \neq 0$), sagittal plane. This case corresponds to the situation of figure 10.8. The $ABCD$ matrix is given by:

\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
\Delta n_e/R & 1
\end{pmatrix},
\] (10.22)

with

\[
\Delta n_e = n_2 \cos \theta_2 - n_1 \cos \theta_1.
\] (10.23)

Notice that with our definition using the reduced slope, all the above elements obey the following relation:

\[
AD - BC = 1.
\] (10.24)

Since the determinant of a product of matrices is the product of the determinants, equation (10.24) holds for any combination of the above elements.
### 10.2.3 Cascaded $ABCD$ systems

Let us now suppose that we want to calculate the evolution of a ray through a system consisting in the combination of elements labeled $1, \ldots, n$ described by their respective $ABCD$ matrices $M_1, \ldots, M_n$. The obvious interest of $ABCD$ matrices is that the matrix $M$ describing the entire system is given by the product of the individual matrices:

$$M = M_n M_{n-1} \cdots M_2 M_1.$$  \hfill (10.25)

### 10.2.4 Transmission of a spherical wave through an $ABCD$ system

Since ray optics and geometrical optics are equivalent, ray optics must provide a convenient way to express the transformations of spherical waves through optical systems. Indeed, spherical waves correspond to the same degree of approximation as geometrical optics. Let us consequently consider the transformation of the spherical wave of figure 10.9 through a system described by an $ABCD$ matrix. This spherical wave is equivalent to many rays characterized by $r(z)$ and $r'(z)$ related by:

$$r'(z) = n_0(z) \frac{dr}{dz} = \frac{n_0(z)r(z)}{R(z)},$$  \hfill (10.26)

where a positive $R(z)$ corresponds to a diverging spherical wave. If we suppose that such a spherical wave with radius of curvature $R_1$ is at the input of a system described by an $ABCD$ matrix (see figure 10.10), then the rays at the output of the system are described by:

$$
\begin{pmatrix}
    r_2 \\
    r'_2
\end{pmatrix} = 
\begin{pmatrix}
    A & B \\
    C & D
\end{pmatrix}
\begin{pmatrix}
    r_1 \\
    r'_1
\end{pmatrix},
$$  \hfill (10.27)
leading to the following radius of curvature for the spherical wave at the output of the system:

\[
\frac{R_2}{n_2} = r_2 = \frac{Ar_1 + Br'_1}{Cr_1 + Dr'_1} = \frac{A(R_1/n_1) + B}{C(R_1/n_1) + D}. \tag{10.28}
\]

Consequently, if we introduce the reduced radius of curvature \( \hat{R} \):

\[
\hat{R}(z) = \frac{R(z)}{n_0(z)}, \tag{10.29}
\]

the paraxial system modifies the spherical wave according to the so-called \( ABCD \) law:

\[
\hat{R}_2 = \frac{A\hat{R}_1 + B}{C\hat{R}_1 + D}. \tag{10.30}
\]

This important law will later be generalized to the eigenmodes of stable laser cavities.

**10.2.5 Evolution of rays in a periodic system**

Let us now consider the evolution of a ray in a periodic system, i.e., a system which reproduces the same sequence of optical elements described by an \( ABCD \) matrix \( \mathbf{M} \). An optical resonator can be modeled by such a periodic system. Indeed, inside the resonator, light rays undergo periodic round-trips, each described by an \( ABCD \) matrix. In such a system, the evolution after a large number of round-trips will depend on the properties of the eigenvalues and eigenvectors of \( \mathbf{M} \).
10.2. RAY OPTICS

10.2.5.1 Eigenrays

Let us call $\lambda_+$ and $\lambda_-$ the eigenvalues of $M$. They are the solutions of the following equation:

$$\begin{vmatrix} A - \lambda & B \\ C & D - \lambda \end{vmatrix} = 0 .$$

(10.31)

Let us introduce the half-trace of the $ABCD$ matrix:

$$m = \frac{A + D}{2} .$$

(10.32)

Then the two solutions are given by

$$\lambda_{\pm} = m \pm \sqrt{m^2 - 1} ,$$

(10.33)

which correspond to two eigenrays $r_+$ and $r_-$. Since any ray $r_0$ at the input of the periodic system can be expanded as a sum of the two eigenrays of the system along:

$$r_0 = c_+ r_+ + c_- r_- ,$$

(10.34)

the ray after $n$ periods of the system will becomes:

$$r_n = c_+ \lambda_+^n r_+ + c_- \lambda_-^n r_- .$$

(10.35)

The behavior of the ray will depend on the values of $\lambda_{\pm}$.

10.2.5.2 Stable system

Let us first suppose that

$$-1 \leq m \leq +1 ,$$

(10.36)

meaning that

$$m^2 = \left( \frac{A + D}{2} \right)^2 \leq 1 .$$

(10.37)

We may then define the angle $\theta$ by:

$$m = \frac{A + D}{2} = \cos \theta ,$$

(10.38)

leading to, according to equation (10.33):

$$\lambda_{\pm} = m \pm i \sqrt{1 - m^2} = \cos \theta \pm i \sin \theta = e^{\pm i \theta} .$$

(10.39)
The eigenvalues of the $ABCD$ matrix are thus complex with a modulus equal to one. Equation (10.35) thus becomes:

$$r_n = c_+ e^{i n \theta} r_+ + c_- e^{-i n \theta} r_- ,$$

(10.40)

which can be rewritten as:

$$r_n = r_0 \cos n \theta + s_0 \sin n \theta ,$$

(10.41)

with $r_0$ given by equation (10.34) and

$$s_0 = i(c_+ r_+ - c_- r_-) .$$

(10.42)

Equation (10.40) shows that the ray oscillates periodically around the system axis, as shown in figure 10.11. This kind of system is called a geometrically stable system: the ray will oscillate periodically around the axis without escaping from it. In the case where the periodic system is an optical cavity, this means that a ray launched inside the cavity will remain inside it and will not escape after a few round-trips.

Figure 10.11: Schematic evolution of a ray in a stable periodic system.

### 10.2.5.3 Unstable system

In the opposite case where

$$|m| > 1 ,$$

(10.43)

meaning that

$$m^2 = \left( \frac{A + D}{2} \right)^2 > 1 ,$$

(10.44)

the eigenvalues of the system become, according to equation (10.33):

$$\lambda_{\pm} = m \pm \sqrt{m^2 - 1} = \left\{ \begin{array}{ll} M \\ 1/M \end{array} \right. ,$$

(10.45)
where $M$ is the transverse magnification of the system. Then the ray after $n$ periods becomes:

\[
r_n = c_+ M^n r_+ + c_- M^{-n} r_-, \quad (10.46)
\]

\[
= r_0 \cosh n\theta + s_0 \sinh n\theta, \quad (10.47)
\]

where $s_0$ again depends on the initial conditions and $\theta = \ln M$.

One can see in this case that $r_n$, and in particular the ray displacement $r_n$, diverges exponentially: a ray launched inside this system will eventually leave it, as shown in figure 10.12. Such a system is said to be geometrically unstable.
Chapter 11
Optical resonators II: Gaussian beams

The preceding chapter has allowed us to introduce the tools originating from geometrical optics that will allow us in the following chapter to calculate the propagation of the modes in stable lasers cavities. In this chapter, we derive these modes by taking into account the diffraction undergone by intracavity light.

11.1 Huygens integral

11.1.1 Paraxial wave equation
Let us consider a monochromatic field of angular frequency $\omega$:

$$E(x, y, z, t) = \mathcal{V}(x, y, z) \ e^{-i\omega t} + \text{c.c.} .$$ (11.1)

Then, in a homogeneous dielectric medium, the Maxwell equations are equivalent to the following wave equation:

$$\Delta \mathcal{V} + k^2 \mathcal{V} = 0 .$$ (11.2)

Let us now suppose that the considered field propagates essentially in the $z$ direction. We can thus write the field in the following way:\footnote{The reader should be careful about those definitions of the amplitudes $A$ and $E$ which are not always consistent with the definitions of the preceding chapters, especially those of chapter 2.}

$$\mathcal{V}(x, y, z) = U(x, y, z) \ e^{ikz} ,$$ (11.3)
which is equivalent to:

\[ E(x, y, z, t) = \mathcal{U}(x, y, z) e^{-i(\omega t - k z)} + \text{c.c.} \]  

(11.4)

The wave equation (11.2) then becomes:

\[ \frac{\partial^2 \mathcal{U}}{\partial x^2} + \frac{\partial^2 \mathcal{U}}{\partial y^2} + \frac{\partial^2 \mathcal{U}}{\partial z^2} + 2i k \frac{\partial \mathcal{U}}{\partial z} = 0 . \]  

(11.5)

If we want to describe a light beam which propagates essentially in the \( +z \) direction, it seems sensible to perform the slowly varying envelope approximation for \( \mathcal{U} \), which reads:

\[ \left| \frac{\partial^2 \mathcal{U}}{\partial z^2} \right| \ll \left| k \frac{\partial \mathcal{U}}{\partial z} \right| , \]  

(11.6)

leading to the \textit{paraxial wave approximation}:

\[ \frac{\partial^2 \mathcal{U}}{\partial x^2} + \frac{\partial^2 \mathcal{U}}{\partial y^2} + 2i k \frac{\partial \mathcal{U}}{\partial z} = 0 , \]  

(11.7)

which can be rewritten in the more general way:

\[ \triangle_{\perp} \mathcal{U} + 2i k \frac{\partial \mathcal{U}}{\partial z} = 0 . \]  

(11.8)

This propagation equation is valid for beams which remain close to the \( z \) axis.

### 11.1.2 Huygens integral in the Fresnel approximation

An alternative to the wave equation is the Huygens integral, issued from the theory of diffraction. Before describing it, let us first explain the Fresnel approximation in the context of spherical waves. Spherical waves are exact solutions of the exact wave equation. At point \( \mathbf{r} \), a spherical wave emitted by a point source located in point \( \mathbf{r}_0 \) reads:

\[ \mathcal{V}(\mathbf{r}; \mathbf{r}_0) = \frac{\exp[ik\rho(\mathbf{r}, \mathbf{r}_0)]}{\rho(\mathbf{r}, \mathbf{r}_0)} , \]  

(11.9)

where \( \rho(\mathbf{r}, \mathbf{r}_0) \) is the distance between the points \( \mathbf{r} \) and \( \mathbf{r}_0 \):

\[ \rho(\mathbf{r}, \mathbf{r}_0) = |\mathbf{r} - \mathbf{r}_0| = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} . \]  

(11.10)

Let us now suppose that these two points are close to the axis \( z \) (see figure
11.1. HUYGENS INTEGRAL

Figure 11.1: Spherical wave close to the z axis.

11.1. The Fresnel approximation consists in expanding $\rho(\mathbf{r}, \mathbf{r}_0)$ at second order in the transverse coordinates:

$$
\rho(\mathbf{r}, \mathbf{r}_0) \simeq z - z_0 + \frac{(x - x_0)^2 + (y - y_0)^2}{2(z - z_0)} .
$$

Equation (11.9) then becomes:

$$
\mathcal{V}(\mathbf{r}; \mathbf{r}_0) \simeq \frac{1}{z - z_0} \exp \left[ i k (z - z_0) + i k \frac{(x - x_0)^2 + (y - y_0)^2}{2(z - z_0)} \right] ,
$$

or, similarly, to:

$$
\mathcal{U}(\mathbf{r}; \mathbf{r}_0) \simeq \frac{1}{z - z_0} \exp \left[ i k \frac{(x - x_0)^2 + (y - y_0)^2}{2(z - z_0)} \right] .
$$

The same kind of approximation can be applied to the Huygens integral. Let us recall that the Huygens principle permits to calculate the field $\mathcal{V}(\mathbf{r})$ at a given point $\mathbf{r}$ from the knowledge of the field $\mathcal{V}_0(\mathbf{r}_0)$ on a given surface $S_0$, by stating that each point $\mathbf{r}_0$ of this surface emits a spherical wavelet with a phase and an amplitude given by the field $\mathcal{V}_0(\mathbf{r}_0)$:

$$
\mathcal{V}(\mathbf{r}) = -\frac{i}{\lambda} \int \int dS_0 \mathcal{V}_0(\mathbf{r}_0) \frac{e^{ik\rho(\mathbf{r}, \mathbf{r}_0)}}{\rho(\mathbf{r}, \mathbf{r}_0)} \cos \theta(\mathbf{r}, \mathbf{r}_0) ,
$$

where $-\frac{i}{\lambda}$ is a normalization factor and $\cos \theta(\mathbf{r}, \mathbf{r}_0)$ is the obliquity factor which depends on the angle $\theta(\mathbf{r}, \mathbf{r}_0)$ between the normal to the surface $S_0$ at $\mathbf{r}_0$ and $\mathbf{r} - \mathbf{r}_0$. Notice that $\lambda$ here is the wavelength in the considered medium. The Fresnel approximation consists in replacing the spherical wave in equation (11.14) by its Fresnel approximation and the obliquity factor by 1, leading to:

$$
\mathcal{V}(x, y, z) = -\frac{ie^{ik(z - z_0)}}{\lambda(z - z_0)} \int \int dxdy_0 \mathcal{V}_0(x_0, y_0, z_0)
\times \exp \left[ i k \frac{(x - x_0)^2 + (y - y_0)^2}{2(z - z_0)} \right] ,
$$

(11.15)
or, similarly, to:

\[
U(x, y, z) = -\frac{i}{\lambda L} \int \int dx_0 dy_0 U_0(x_0, y_0, z_0) \exp \left[ i k \frac{(x-x_0)^2 + (y-y_0)^2}{2L} \right],
\]

(11.16)

with \( L = z - z_0 \).

In one dimension, this Huygens integral at the Fresnel approximation becomes:

\[
U(x, z) = \sqrt{-\frac{i}{\lambda L}} \int dx_0 U_0(x_0, z_0) \exp \left[ i k \frac{(x-x_0)^2}{2L} \right].
\]

(11.17)

In all these expressions, one should be conscious of the fact that \( \lambda \) is the wavelength in the considered medium of refractive index \( n_0 \), namely:

\[
\lambda = \frac{\lambda_0}{n_0}.
\]

(11.18)

11.2 Gaussian beams

11.2.1 Complex spherical wave

As we have just seen, the field created at a point \((x, y, z)\) by a point source of spherical wave located in \((x_0, y_0, z_0)\) reads:

\[
U(x, y, z) = \frac{1}{z - z_0} \exp \left[ i k \frac{(x-x_0)^2 + (y-y_0)^2}{2(z-z_0)} \right] = \frac{1}{R(z)} \exp \left[ i k \frac{(x-x_0)^2 + (y-y_0)^2}{2R(z)} \right],
\]

(11.19)

with \( R(z) = z - z_0 \) and where we do not take care of the amplitude of the wave. This can be generalized to any origin value of the curvature of the wavefront by taking:

\[
R(z) = R_0 + z - z_0.
\]

(11.20)

Notice that with our convention, \( R > 0 \) corresponds to a diverging spherical wave and \( R < 0 \) to a converging spherical wave.

The spherical wave of equation (11.19) obeys the paraxial wave equation (11.8) and obeys the Huygens-Fresnel integral (11.16) for any values of \( x_0, y_0, \) and \( z_0 \). These three numbers can be considered as simple parameters of the wave. In particular, there is no reason why we could not take them
complex. Let us try to take \( x_0 = y_0 = 0 \) and replace \( z_0 - q_0 \) with \( q_0 \) complex. Then equation (11.20) must be replaced by:

\[
q(z) = z - (z_0 - q_0) = q_0 + z - z_0 ,
\]

which is also complex. In particular, we have:

\[
q(z_0) = q_0 .
\]

Equation (11.19) then becomes:

\[
\mathcal{U}(x, y, z) = \frac{1}{z - z_0 + q_0} \exp \left[ ik \frac{x^2 + y^2}{2(z - z_0 + q_0)} \right] = \frac{1}{q(z)} \exp \left[ ik \frac{x^2 + y^2}{2q(z)} \right].
\]

(11.23)

\( q(z) \) is called the complex radius of curvature of the beam. Let us separate the real and imaginary parts of \( 1/q(z) \) by defining \( R(z) \) and \( w(z) \) according to:

\[
\frac{1}{q(z)} = \frac{1}{R(z)} + i \frac{\lambda}{\pi w(z)^2} .
\]

(11.24)

Then equation (11.23) becomes:

\[
\mathcal{U}(x, y, z) = \frac{1}{q(z)} \exp \left[ ik \frac{x^2 + y^2}{2R(z)} - \frac{x^2 + y^2}{w(z)^2} \right].
\]

(11.25)

### 11.2.2 Higher-order Gaussian modes

The Gaussian beam of equation (11.25) is only the lowest-order member of an infinite family of solutions of the Huygens-Fresnel integral. Two main different families will be described here: the Hermite-Gaussian modes in rectangular coordinates and the Laguerre-Gaussian modes in cylindrical coordinates.

#### 11.2.2.1 Formal derivation of the lowest-order mode

Let us re-derive the lowest-order mode in a more formal manner. We are looking for a solution of the type:

\[
\mathcal{U}(x, y, z) = A(z) \exp \left( ik \frac{x^2 + y^2}{2q(z)} \right).
\]

(11.26)

We inject equation (11.26) into the paraxial wave equation (11.8), leading to:

\[
\frac{k^2}{q(z)^2} \left( -1 + \frac{d q}{dz} \right) (x^2 + y^2) + 2ik \left( \frac{A(z)}{q(z)} + \frac{d A}{dz} \right) = 0 ,
\]

(11.27)
leading to the following set of differential equations:

\[
\frac{dq}{dz} = 1, \quad (11.28) \\
\frac{1}{A(z)} \frac{dA}{dz} = -\frac{1}{q(z)}, \quad (11.29)
\]

which have the following solutions:

\[
q(z) = q_0 + z - z_0, \quad (11.30) \\
A(z) = \frac{q_0}{q(z)} \cdot (11.31)
\]

This is equivalent to the result of equation (11.25).

### 11.2.2.2 Higher-order modes in rectangular coordinates

In rectangular coordinates, one can separate the two transverse variables and look for solutions of the type:

\[
U_{nm}(x, y, z) = U_n(x, z) U_m(y, z), \quad (11.32)
\]

each component satisfying a 2D paraxial wave equation:

\[
\frac{\partial^2 U_n}{\partial x^2} + 2ik \frac{\partial U_n}{\partial z} = 0. \quad (11.33)
\]

The solution can be sought in the following form:

\[
U_n(x, z) = A[ q(z) ] \left( \frac{x}{p(z)} \right) \exp \left( ik \frac{x^2}{2q(z)} \right) . \quad (11.34)
\]

After long and tedious calculations, one finds that the following so-called Hermite-gaussian mode is a solution:

\[
U_n(x, z) = \left( \frac{2}{\pi} \right)^{1/4} \left( \frac{1}{2^n n! w_0} \right)^{1/2} \left( \frac{q_0}{q(z)} \right)^{1/2} \left( \frac{q_0 q^*(z)}{q(z)} \right)^{n/2} \\
\times H_n \left( \sqrt{2x} \frac{w(z)}{w(z)} \right) \exp \left[ ik \frac{x^2}{2q(z)} \right], \quad (11.35)
\]

where \( q(z) \) is still given by equation (11.30) and \( H_n \) is the Hermite polynomial of integer order \( n \). By defining the Gouy phase shift

\[
\tan \psi(z) \equiv -\frac{\pi w(z)^2}{R(z) \lambda}, \quad (11.36)
\]
equation (11.35) becomes:

\[
U_n(x, z) = \left( \frac{2}{\pi} \right)^{1/4} \sqrt{\frac{\exp[i(2n + 1)(\psi(z) - \psi_0)]}{2^n n! w(z)}} 
\times H_n \left( \frac{\sqrt{2}x}{w(z)} \right) \exp \left[ \frac{i k x^2}{2q(z)} \right].
\]  

(11.37)

These Hermite-Gaussian modes constitute a complete basis set of orthonormal functions.

The fact that we have separated the \(x\) and \(y\) coordinates makes this formalism particularly suitable for the treatment of astigmatic systems. In this case, the two directions must be treated independently, the overall field being the product of the 2D fields along the \(x\) and \(y\) directions.

11.2.2.3 Higher-order modes in cylindrical coordinates

In cylindrical coordinates, the so-called Laguerre-Gaussian modes constitute an alternative family of solutions given by:

\[
U_{pm}(r, \theta, z) = \sqrt{\frac{2p!}{(1 + \delta_{0m})\pi(m + p)!}} \frac{\exp[i(2p + m + 1)(\psi(z) - \psi_0)]}{w(z)} 
\times \left( \frac{\sqrt{2r}}{w(z)} \right)^m L_p^m \left( \frac{2r^2}{w(z)^2} \right) \exp \left[ \frac{i k r^2}{2q(z)} - im\theta \right].
\]  

(11.38)

In this expression, the integer \(p \geq 0\) is the radial index and the integer \(m\) is the azimuthal mode index. The \(L_p^m\) are the generalized Laguerre polynomials.

11.3 Physical properties of Gaussian beams

11.3.1 Fundamental mode

The lowest-order Gaussian mode is called the TEM\(_{00}\) mode. If we take \(z_0 = 0\), its field propagates according to:

\[
U(x, y, z) = U_0 \frac{w_0}{w(z)} e^{i\psi(z)} \exp \left[ \frac{ik}{2R(z)} x^2 + \frac{i}{2} \frac{\lambda}{\pi w(z)^2} \right],
\]  

(11.39)

with

\[
\frac{1}{q(z)} = \frac{1}{R(z)} + i \frac{\lambda}{\pi w(z)^2}.
\]  

(11.40)
In free space, the complex radius of curvature \( q(z) \) propagates according to:

\[
q(z) = z - i z_R , \tag{11.41}
\]

with the following definition for the so-called Rayleigh range \( z_R \):

\[
z_R = \frac{\pi w_0^2}{\lambda} . \tag{11.42}
\]

The parameters of the beam evolve in the following manner during propagation:

\[
w(z) = w_0 \sqrt{1 + \left( \frac{z}{z_R} \right)^2} , \tag{11.43}
\]

\[
R(z) = z + \frac{z_R^2}{z} , \tag{11.44}
\]

\[
\psi(z) = -\tan^{-1}\left( \frac{z}{z_R} \right) . \tag{11.45}
\]

The evolution of the Gaussian beam during its propagation along \( z \) is schematized in figure 11.2. In any plane \( z \), the beam is characterized by the radius \( w(z) \) at \( 1/e^2 \) of its Gaussian transverse intensity distribution and the radius of curvature \( R(z) \) of its spherical wavefront. Their characteristics are derived from equations (11.43) and (11.44) and their evolutions versus \( z \) are shown in figure 11.3.

Two different regions must be distinguished. The near-field region \( |z| < z_R \) is called the Rayleigh zone. In this region, \( w \) remains of the order of \( w_0 \); the beam hardly experiences any spreading due to diffraction. The minimum
11.3. PROPERTIES OF GAUSSIAN BEAMS

Figure 11.3: Evolutions of (a) $w/w_0$ and (b) $R/z_R$ versus $z/z_R$. The dashed lines correspond to the asymptotic behavior for $z \to \pm \infty$.

value of the beam radius is $w_0$, reached at $z = 0$. The plane $z = 0$ is called the beam waist, and the value $w_0$ is called the beam radius at the waist or, shortly, the waist. In the Rayleigh zone, the wavefront is initially flat and reaches its maximum curvature in $z = z_R$.

The far-field region corresponds to $|z| \gg z_R$. In this region, the wavefront is essentially that of a spherical wave centered in $z = 0$, which tends to a plane wavefront for large values of $z$. The beam radius $w(z)$ evolves linearly with $z$. The divergence angle of the beam becomes:

$$\theta_{1/e} = \frac{\lambda}{\pi w_0}. \tag{11.46}$$

11.3.2 Hermite-Gaussian modes

The Hermite-Gaussian TEM$_{mn}$ mode of equation (11.35) has the same spherical wavefront as the fundamental TEM$_{00}$ but the following intensity distribution:

$$I_{mn}(x, y, z) \propto \left[ H_m \left( \frac{\sqrt{2} x}{w(z)} \right) H_n \left( \frac{\sqrt{2} y}{w(z)} \right) \right]^2 \exp \left[- \frac{2(x^2 + y^2)}{w(z)^2} \right]. \tag{11.47}$$

Some examples of intensity profiles of these modes are shown in figure 11.4. The mode of order $m$ corresponds to the $m^{th}$ Hermite polynomial and consequently exhibits $m$ intensity nodes.

11.3.3 Laguerre-Gaussian modes

In cylindrical coordinates, the Laguerre-Gaussian mode of indices $pm$ has $p$ nodes in the radial direction and $m$ nodal planes in the orthoradial direction.
Figure 11.4: Intensity profiles of a few Hermite-Gaussian beams.
11.3. PROPERTIES OF GAUSSIAN BEAMS

Figure 11.5: Intensity profiles of a few Laguerre-Gaussian beams.

Some examples of intensity distributions are shown in figure 11.5. The last index $j$ in figure 11.5 corresponds to the choice between a $\cos m\theta$ and a $\sin m\theta$ angular variation of the field.
Chapter 12

Optical resonators III: Stable cavity modes

In the preceding chapter, we have described the solutions of the paraxial wave equation in a homogeneous medium. Such modes are generated inside laser cavities. The question which arises now is: what are the characteristics of the eigenmodes of a given cavity? To answer this question, we start with the simple case of a two-mirror cavity and then generalize this problem using the $ABCD$ formalism of chapter 10 applied to the Gaussian beams of chapter 11.

12.1 Stable two-mirror cavities

Let us consider the cavity of figure 12.1. It is built with two mirrors $M_1$ and $M_2$ of respective radii of curvature $R_1$ and $R_2$, separated by a distance $L$. Notice that according to our definition, the cavity length is $L_{\text{cav}} = 2L$. We take $R_1$ and $R_2$ positive when the two mirrors are concave, as shown in the example of figure 12.1.

12.1.1 Derivation of the mode

The question we want to answer here is: is there a Gaussian beam that can be an eigenmode of this cavity? Or, in other words, is there a Gaussian beam that can propagate through one round-trip inside this cavity and lead back to the initial Gaussian beam. We can see that such a solution can exist if the curvature of the wavefronts on the two mirrors can match the radii of curvature of the mirrors. In other words, if the origin $z = 0$ corresponds to the waist of this Gaussian beam, we must have, according to equation
Figure 12.1: Two-mirror cavity.

(11.44):

\[-R_1 = R(z_1) = z_1 + \frac{z_R^2}{z_1}, \quad (12.1)\]

\[R_2 = R(z_2) = z_2 + \frac{z_R^2}{z_2}, \quad (12.2)\]

where $z_R$ is the Rayleigh range of the beam we are trying to determine and $z_1$ and $z_2$ are the abscissa of the mirrors with respect to the waist located in $z = 0$. They obey the following equation:

\[L = z_2 - z_1. \quad (12.3)\]

If we introduce the following “$g$ parameters”:

\[g_1 = 1 - L/R_1 \text{ and } g_2 = 1 - L/R_2, \quad (12.4)\]

the solutions of equations (12.1-12.3) are given by:

\[z_R^2 = \frac{g_1 g_2 (1 - g_1 g_2)}{(g_1 + g_2 - 2 g_1 g_2)^2} L^2, \quad (12.5)\]

and

\[z_1 = -\frac{g_2(1 - g_1)}{g_1 + g_2 - 2 g_1 g_2} L, \quad (12.6)\]

\[z_2 = \frac{g_1 (1 - g_2)}{g_1 + g_2 - 2 g_1 g_2} L. \quad (12.7)\]

This leads to the following expression for the beam waist:

\[w_0^2 = \frac{\lambda L}{\pi} \sqrt{\frac{g_1 g_2 (1 - g_1 g_2)}{(g_1 + g_2 - 2 g_1 g_2)^2}}, \quad (12.8)\]
and to the following expressions for the beam radii on the two mirrors:

\[ w_1^2 = \frac{\lambda L}{\pi} \sqrt{\frac{g_2}{g_1(1 - g_1g_2)}}, \]
\[ w_2^2 = \frac{\lambda L}{\pi} \sqrt{\frac{g_1}{g_2(1 - g_1g_2)}}. \]

12.1.2 Stability diagram

One can see from equations (12.5), (12.9), and (12.10) that such a Gaussian beam solution exists only if

\[ 0 \leq g_1g_2 \leq 1. \]  

(12.11)

This is the stability condition for the two-mirror resonator. It corresponds to the shaded area in the \((g_1, g_2)\) representation of figure 12.2. The points located inside the shaded area of figure 12.2 correspond to geometrically stable resonators (according to the definition given in section 10.2) for which one can apply the equations derived in the present section. The points located outside this area correspond to cavities which also have modes. But the modes of such unstable resonators are not Gaussian modes.
Some particular cases of two-mirror resonators have particular names. The case where $R_1 = R_2$ is called the symmetric resonator. The case where mirror $M_1$ is planar ($g_1 = 1$) is called the half-symmetric resonator. The case where $R_1 = R_2 = L$ is called the symmetric confocal resonator. Indeed, in this case, the focal points of the two mirrors coincide in the middle of the cavity. The case where $R_1, R_2 \gg L$ is called the near-planar resonator. Finally the case where $R_1 + R_2$ is almost equal to $L$ is called the near-concentric resonator and the hemispherical resonator corresponds to the case where $M_1$ is planar and $R_2$ is slightly larger than $L$.

As an example, let us consider the situation in which mirror $M_1$ is planar, leading to $g_1 = 1$ and let us take $R_2 = R$. Then, the waist is located on the planar mirror (see figure 12.3(a)) and the beam radii on the two mirrors are given by:

$$w_1 = w_0 = \sqrt{\frac{\lambda}{\pi}} \sqrt{L(R-L)} ,$$  \hspace{1cm} (12.12)

$$w_2 = \sqrt{\frac{\lambda R}{\pi}} \sqrt{\frac{L}{R-L}} .$$  \hspace{1cm} (12.13)

Figure 12.3(b) represents the evolution of these two beam radii versus the cavity length $L$ for $\lambda = 1 \ \mu m$ and $R = 1 \ m$. We can see that close to the stability limit $L = R$, the beam waist goes to zero while the beam radius on the spherical mirror tends to infinity.

### 12.1.3 Frequencies of the transverse modes

We have considered up to now only the lowest-order Gaussian mode. If we turn to higher-order Hermite-Gaussian modes, equation (11.37) shows that
the phase accumulated by the TEM\textsubscript{nm} mode from mirror \( M_1 \) to mirror \( M_2 \) is:

\[
\varphi(1 \rightarrow 2) = kL + (n + m + 1) [\psi(z_2) - \psi(z_1)]
\]

where \( \psi(z) \) is the Gouy phase shift given in equation (11.45). Using equations (12.6) and (12.7), one can show that

\[
\psi(z_2) - \psi(z_1) = - \cos^{-1} \pm \sqrt{g_1 g_2},
\]

where the + sign applies to the upper right quadrant of figure 12.2 \((g_1, g_2 > 0)\) and the minus sign to the lower left quadrant of this diagram \((g_1, g_2 < 0)\). If we want the TEM\textsubscript{nm} mode to be resonant in our cavity, then the phase of equation (12.14) must be equal to an integer number of times \( \pi \), leading to:

\[
\frac{\omega}{c}L - (n + m + 1) \cos^{-1} \pm \sqrt{g_1 g_2} = p\pi,
\]

where \( p \) is an integer. This leads to the following resonance frequencies for the transverse modes:

\[
\frac{\omega_{\text{trans}}}{2\pi} = \frac{c}{2L} \left[ p + (n + m + 1) \frac{\cos^{-1} \pm \sqrt{g_1 g_2}}{\pi} \right].
\]

The term due to the Gouy phase shift evolves between 0 and 1:

\[
\cos^{-1} \pm \sqrt{g_1 g_2} = \begin{cases} 
0 & \text{for the near-planar cavity \((g_1, g_2 \to 1)\)}, \\
1/2 & \text{for the near-confocal cavity \((g_1, g_2 \to 0)\)}, \\
1 & \text{for the near-concentric cavity \((g_1, g_2 \to -1)\)}.
\end{cases}
\]

The frequency difference between two successive tranverse modes is:

\[
\Delta \nu_{\text{trans}} = \frac{c}{2L} \frac{\cos^{-1} \pm \sqrt{g_1 g_2}}{\pi}.
\]

Figure 12.4 represents the transverse mode spectra of three different types of cavities. The confocal cavity is a particularly puzzling situation in which the even and odd modes are grouped in two combs of eigenfrequencies separated by one half of the cavity free spectral range.

### 12.2 ABCD matrices and Gaussian beams

As we have just seen, the case of the two-mirror cavity can be simply dealt with by applying the laws of propagation of Gaussian beams derived in chapter 11. However, this direct method becomes too cumbersome in the general case of more complicated resonators. A more powerful formalism, based on the ABCD matrices of chapter 10, can then be used, as we are going to see now.
Figure 12.4: Spectrum of the transverse modes of three types of cavities. The numbers 0...4 labeling the transverse modes correspond to $n + m$. 
12.2. ABCD FORMALISM

12.2.1 Generalization of the Huygens integral to any paraxial system

As we have seen in section 11.2, in the case of only one transverse dimension $x$, the Huygens integral can be used to obtain the field distribution $V_2$ in plane $z_2$ from the field distribution $V_1$ in plane $z_1$ where planes $z_1$ and $z_2$ are separated by a distance $L$ of propagation in a homogeneous medium:

$$V_2(x_2) = e^{ikL} \int_{-\infty}^{\infty} K(x_2, x_1)V_1(x_1)dx_1 = \sqrt{-\frac{i}{\lambda L}} \int_{-\infty}^{\infty} V_1(x_1)e^{ik\rho(x_1, x_2)}dx_1,$$  \hspace{1cm} (12.20)

where the path length $\rho(x_1, x_2)$ is given, in the Fresnel approximation, by:

$$\rho(x_1, x_2) = \sqrt{L^2 + (x_2 - x_1)^2} \approx L + \frac{(x_2 - x_1)^2}{2L}. \hspace{1cm} (12.21)$$

The Huygens-Fresnel kernel for free space propagation is thus:

$$K(x_2, x_1) = \sqrt{-\frac{i}{\lambda L}} \exp \left[ \frac{\pi(x_2 - x_1)^2}{\lambda L} \right]. \hspace{1cm} (12.22)$$

We want to generalize this Huygens-Fresnel kernel to the situation in which the incident wave $V_1(x_1)$ is sent through any paraxial system characterized by its $ABCD$ matrix (see figure 12.5). This system can itself be the combination of individual paraxial systems (lenses, free space propagation, mirrors, interfaces,...). If we consider a ray entering the $ABCD$ system at...
12. OPTICAL RESONATORS III

Figure 12.6: Propagation of a ray from \( P_1 \) to \( P_2 \) which are conjugate points through the paraxial system described by matrix \( ABCD \) (Taken from A. E. Siegman, Lasers, op. cit.).

position \( x_1 \) and exiting from it at position \( x_2 \), then, since \( x_2 = Ax_1 + Bx'_1 \), we have:

\[
x'_1 = \frac{x_2 - Ax_1}{B},
\]

(12.23)

leading to:

\[
x'_2 = Cx_1 + Dx'_1 = \frac{Dx_2 - x_1}{B},
\]

(12.24)

where we have used the fact that the determinant of the \( ABCD \) matrix is equal to 1. The ray entering the system at \( x_1 \) with the reduced slope \( x'_1 \) can be seen as emerging from a point source \( P_1 \) located at a distance \( R_1 \) before the entrance plane of the \( ABCD \) system (see figure 12.6) with:

\[
\frac{R_1}{n_1} = \frac{x_1}{x'_1} = \frac{Bx_1}{x_2 - Ax_1}.
\]

(12.25)

Similarly, at the exit, the ray can be seen as a spherical wave of radius of curvature \( R_2 \) given by:

\[
\frac{R_2}{n_2} = \frac{x_2}{x'_2} = \frac{Bx_2}{Dx_2 - x_1}.
\]

(12.26)

The centers \( P_1 \) and \( P_2 \) of the spherical waves are conjugate points: any ray emerging from \( P_1 \) must arrive in \( P_2 \). According to Fermat’s principle, all the rays connecting two conjugate points must have the same optical length. In particular, the ray going from \( P_1 \) to \( P_2 \) via \( X_1 \) and \( X_2 \) that we consider here (see figure 12.6) must have the same optical length as the one going from \( P_1 \) to \( P_2 \) along the optical axis. Let us call \( L_0 \) the optical thickness, i. e., the
optical length along the optical axis, of our \(ABCD\) system. It is given by:

\[ L_0 = \sum n_i L_i , \]  

(12.27)

where the \(n_i\)'s and \(L_i\)'s are the indices and thicknesses of the different elements constituting the considered \(ABCD\) system. Then the optical length of the ray connecting \(P_1\) to \(P_2\) along the axis is:

\[ P_1P_2 = n_1R_1 + L_0 - n_2R_2 . \]  

(12.28)

The optical length of the ray going from \(P_1\) to \(P_2\) via \(X_1\) and \(X_2\) is given, in the Fresnel approximation, by:

\[ P_1X_1X_2P_2 \simeq n_1 \left( R_1 + \frac{x_1^2}{2R_1} \right) + \rho(x_1, x_2) - n_2 \left( R_2 + \frac{x_2^2}{2R_2} \right) , \]  

(12.29)

Combining equations (12.25), (12.26), (12.28), and (12.29), we obtain:

\[ \rho(x_1, x_2) = L_0 + \frac{Ax_1^2 - 2x_1x_2 + Dx_2^2}{2B} , \]  

(12.30)

which is the generalization of equation (12.21) to any paraxial system described by \(ABCD\). We eventually obtain the following formulation of the generalized Huygens-Fresnel integral:

\[ \mathcal{V}_2(x_2) = e^{ikL_0} \int_{-\infty}^{\infty} K(x_2, x_1) \mathcal{V}_1(x_1) dx_1 , \]  

(12.31)

with the following kernel:

\[ K(x_2, x_1) = \sqrt{-\frac{\lambda_0}{B\lambda_0}} \exp \left[ i \frac{\pi}{B\lambda_0} (Ax_1^2 - 2x_1x_2 + Dx_2^2) \right] , \]  

(12.32)

where \(\lambda_0\) is the wavelength in vacuum. The scale factor \(\sqrt{-\frac{\lambda_0}{B\lambda_0}}\) has been introduced to conserve power and to make the generalized Huygens-Fresnel kernel coincide with the one of equation (12.22) in the case of free-space propagation.

**Comment:** One notices that the exchange of \(x_1\) and \(x_2\) in the propagation kernel leads to the exchange of elements \(A\) and \(D\) in the \(ABCD\) matrix, allowing to immediately deduce the matrix in the reverse propagation direction from the knowledge of the \(ABCD\) matrix in the forward propagation direction:

\[ \left( \begin{array}{cc} A & B \\ C & D \end{array} \right)_{\text{propagation along } -z} = \left( \begin{array}{cc} D & B \\ C & A \end{array} \right) . \]  

(12.33)
12.2.2 The \textit{ABCD} formalism for Gaussian beams

Let us now imagine that we launch a Gaussian beam on our \textit{ABCD} paraxial system. For one transverse dimension, the input beam reads:

\[
\mathcal{U}_1(x_1) = \exp \left( \frac{i \pi x_1^2}{q_1 \lambda_1} \right),
\]

with

\[
1 = \frac{1}{R_1} + i \frac{\lambda_1}{\pi w_1^2}.
\]

Notice that the wavelength is always the wavelength in the considered medium. For example, \(\lambda_1\) in equations (12.34) and (12.35) is the wavelength in the medium of refractive index \(n_1\) located at the input of the considered \textit{ABCD} system. Using the generalized Huygens-Fresnel principle of equations (12.31) and (12.32), the beam at the output of the paraxial system is:

\[
\mathcal{U}_2(x_2) = \sqrt{-\frac{i}{B \lambda_0}} \int_{-\infty}^{\infty} \exp \left[ i \frac{\pi x_1^2}{q_1 \lambda_1} + i \frac{\pi}{B \lambda_0} (Ax_1^2 - 2x_1x_2 + Dx_2^2) \right] dx_1.
\]

Using equation (9.6) and after a few lines of calculations, \(\mathcal{U}_2(x_2)\) can be written:

\[
\mathcal{U}_2(x_2) = \sqrt{\frac{1}{A + n_1 B / q_1}} \exp \left( i \frac{\pi x_2^2}{q_2 \lambda_2} \right),
\]

with

\[
q_2 = \frac{A(q_1/n_1) + B}{C(q_1/n_1) + D}.
\]

Equations (12.37) and (12.38) show that a paraxial system transforms a Gaussian beam into another Gaussian beam, with the two complex radii of curvature related thanks to equation (12.38). This equation is called the \textit{ABCD law for Gaussian beams}.

In a manner similar to equation (10.30) we define the reduced complex radius of curvature \(\hat{q}\) of the Gaussian beam by:

\[
\frac{1}{\hat{q}} = \frac{n_0}{q} = \frac{n_0}{R} + i \frac{n_0 \lambda}{\pi w^2} = \frac{1}{R} + i \frac{\lambda}{\pi w^2}.
\]

Then, the \textit{ABCD} law (12.38) can be written as

\[
\hat{q}_2 = \frac{Aq_1 + B}{Cq_1 + D}.
\]
12.2.3 Application to cavities

This $ABCD$ formalism can be used to determine the Gaussian eigenmode of any linear or planar cavity. Let us consider the cavity schematized in figure 12.7. One round-trip inside this cavity starting from a given reference plane is described by a given $ABCD$ matrix which can be calculated by multiplying the $ABCD$ matrices of the individual intracavity elements. In the case of a planar ring cavity, the spherical mirrors at non normal incidence create astigmatism and the $ABCD$ matrices are different in the sagittal and tangential planes. Consequently, one must determine independently the Gaussian mode in the tangential and sagittal planes, the overall mode being the product of the two one-transverse-dimension beams.

Let us thus look for the Gaussian eigenmode of the cavity of figure 12.7. The Gaussian beam in the reference plane is described by its complex reduced radius of curvature $\hat{q}$ which must verify:

$$\hat{q} = \frac{A\hat{q} + B}{C\hat{q} + D}.$$  \hfill (12.41)

This is equivalent to:

$$\frac{1}{\hat{q}^2} + \frac{A - D}{B} \frac{1}{\hat{q}} + \frac{1 - AD}{B^2} = 0,$$  \hfill (12.42)
which has the following solutions:

\[
\frac{1}{q^*} = \frac{D - A}{2B} \pm \frac{1}{B} \sqrt{\left(\frac{A + D}{2}\right)^2 - 1}.
\] (12.43)

If we want this solution to correspond to a Gaussian beam, it must be complex. This imposes the following stability condition:

\[
\text{Stable cavity } \iff -1 \leq m = \frac{A + D}{2} \leq 1,
\] (12.44)

which is equivalent to the stability condition found at the end of chapter 10 for periodic systems [see equation (10.36)].

In the case of a stable cavity, equation (12.43) becomes:

\[
\frac{1}{q^*} = \frac{D - A}{2B} + \frac{i}{|B|} \sqrt{1 - m^2},
\] (12.45)

leading, using equation(12.39), to the following mode characteristics:

\[
R = \frac{2n_0 B}{D - A},
\] (12.46)

and

\[
w = \sqrt{\frac{\lambda_0 |B|}{\pi}} \frac{1}{\sqrt{1 - m^2}}.
\] (12.47)

### 12.3 Mode losses

Up to now, we have derived the cavity modes without taking the sizes of the mirrors into account. We have seen that, for a stable cavity, the characteristics of the Hermite-Gaussian and Laguerre-Gaussian modes are independent of the mirror sizes. Moreover, these modes have been shown to exhibit no diffraction losses. This is of course no longer true when the mirror size \(2a\) is no longer large compared with the mode diameter \(2w\).

Let us consider for example a two-mirror resonator of length \(L\), with one mirror, say \(M_1\), of finite size. We call \(2a\) the transverse width of this mirror in the \(x\) direction for a strip resonator, or the diameter of the mirror for a circularly symmetric resonator. We then define the resonator Fresnel number \(N_f\) according to:

\[
N_f = \frac{a^2}{L \lambda}.
\] (12.48)
This parameter is the number of Fresnel zones across mirror $M_1$ as seen from mirror $M_2$. In the case of a symmetric confocal resonator of length $L$, equation (12.9) leads to:

$$w_1 = \sqrt{\frac{\lambda L}{\pi}}.$$ \hspace{1cm} (12.49)

In this case, $\pi N_f$ is simply the ratio of the resonator mirror area to the mode area:

$$\pi N_f = \frac{\pi a^2}{\pi w_1^2}.$$ \hspace{1cm} (12.50)

This equality remains valid, to a numerical factor, for other types of resonators.

For higher order modes, the radius $s_n$ of the mode evolves roughly as the square root of the mode order $n$:

$$s_n \approx \sqrt{n} w_1 \approx \sqrt{\frac{nL\lambda}{\pi}}.$$ \hspace{1cm} (12.51)

Consequently, the largest-order Hermite-Gaussian or Laguerre-Gaussian mode that can still fit within the aperture of width or diameter $2a$ is roughly given by the following order:

$$n_{\text{max}} \approx \frac{a^2}{w_1^2} = \pi N_f.$$ \hspace{1cm} (12.52)

The Fresnel number $N_f$ is consequently a measure of the number of transverse modes that can oscillate in a given resonator.
Chapter 13

Optical resonators IV: Unstable cavities

The modes of the geometrically stable resonators that we have considered in the preceding chapter are independent of the sizes of the mirrors, provided these mirrors are large enough. When the sizes of the mirrors are large compared with the mode radii on these mirrors, the finite mirror sizes create diffraction losses which can often be considered as small perturbations to the modes which remain essentially the ones that we have determined in chapter 12. The situation is of course completely different in the case of geometrically unstable resonators. In this chapter, we give a short introduction to such cavities, following first geometric optics (see section 13.1) followed by an analysis of some effects of Huygens-Fresnel diffraction by the finite size optical elements (see section 13.2).

13.1 Geometrical analysis of unstable resonators

Let us recall that for an unstable resonator (see subsection 10.2.5), the half-trace $m$ of the $ABCD$ matrix for one round-trip verifies $|m| > 1$ and that the eigenvalues of this matrix are:

$$\lambda_a, \lambda_b = m \pm \sqrt{m^2 - 1} = M, \frac{1}{M},$$

(13.1)

where $M$, whose modulus is also larger than 1, is called the transverse magnification of the cavity. If $m > 1$ (resp. $m < -1$), the cavity is said to be a positive branch (resp. negative branch) unstable resonator.
13. OPTICAL RESONATORS IV

13.1.1 Unstable resonator eigenwaves

If we apply the $ABCD$ formalism of chapter 12 to our resonator, than the eigensolutions given by equation (12.43) are both real and read (we suppose in this entire chapter that the refractive index is $n_0 = 1$):

$$\frac{1}{q_a}, \frac{1}{q_b} = \frac{D - A}{2B} \pm \frac{\sqrt{m^2 - 1}}{B} = \frac{1}{R_a}, \frac{1}{R_b}. \quad (13.2)$$

These formal solutions are consequently unbounded spherical waves (without any Gaussian amplitude profile) whose radii of curvature are given by:

$$\frac{1}{R_a}, \frac{1}{R_b} = \frac{D - \lambda_a}{B}, \frac{D - \lambda_b}{B}. \quad (13.3)$$

13.1.2 Positive branch unstable resonators

In the case where $m$ is larger than 1, equation (13.1) becomes:

$$\lambda_a = m + \sqrt{m^2 - 1} = M, \quad (13.4)$$
$$\lambda_b = m - \sqrt{m^2 - 1} = 1/M, \quad (13.5)$$

where, in this case, $M > 1$. The two corresponding eigenwaves have their radii of curvature given by equations (13.2) and (13.3). Let us consider a ray $r_{a(0)}$ which is perpendicular to the surface of the eigenwave characterized by $R_a$. Then the position and the slope of this ray verify:

$$r_{a(0)}' = \frac{r_{a(0)}}{R_a}. \quad (13.6)$$

After one round-trip inside the cavity, the position of the ray will be given by:

$$r_{a(1)} = Ar_{a(0)} + Br_{a(0)}' = (A + B \frac{1}{R_a}) r_{a(0)} = \frac{r_{a(0)}}{\lambda_a} = \frac{r_{a(0)}}{M}, \quad (13.7)$$

showing that the solution corresponding to $R_a$ is the demagnifying eigenwave.

Conversely, a ray $r_{b(0)}$ which is perpendicular to the surface of the eigenwave characterized by $R_b$ becomes, after one cavity round-trip:

$$r_{b(1)} = \frac{r_{b(0)}}{\lambda_b} = M r_{b(0)}, \quad (13.8)$$

showing that $r_{b(0)}$ corresponds to the so-called magnifying eigenwave.

Suppose now that we launch into the resonator a beam (with a finite size) that has a wavefront radius of curvature given by $R_a$ or $R_b$. Then, depending on the solution, the size of the beam will either magnify or demagnify in size by the ratio $M$ or $1/M$ at each round-trip through the cavity, as shown in figure 13.1.
13.1.3 Negative branch unstable resonators

Suppose now that we consider a negative branch unstable resonator ($m < -1$). Then the two eigenvalues are:

\[
\begin{align*}
\lambda_a &= -|m| + \sqrt{m^2 - 1} = 1/M , \\
\lambda_b &= -|m| - \sqrt{m^2 - 1} = M ,
\end{align*}
\]

where $M$ is now negative and verifies $M < -1$. Then the solution corresponding to $R_a$ is the magnifying eigenwave while the solution corresponding to $R_b$ is the demagnifying eigenwave, as shown in figure 13.2.

13.1.4 Real unstable resonators

As we have just seen, for both positive and negative branch unstable resonators, there exists a magnifying eigenwave and a demagnifying eigenwave. As we have seen in chapter 10, this is the very definition of a geometrically unstable resonator. One important difference between the modes of positive and negative branch resonators is that because of the inversion at each round-trip, the modes of negative branch resonators exhibit an odd number
of focii per round-trip. Notice also that since the magnifying solution is the only stable solution (in the sense of the stability of solutions of dynamical systems), it will be the one that will provide the actual solution that will oscillate in the cavity. In the case of a “real” laser cavity, the geometric mode will usually be bounded by the smallest diameter element inside the cavity, as shown for example in figure 13.3 for the so-called hard-edged confocal unstable resonators.

13.1.5 Geometrical output coupling value

According to a pure geometric analysis, we have just seen that at each round-trip the intracavity field expands transversally by the factor $M$. If we consider a strip resonator, i.e., a one-dimensional resonator, this means that the field amplitude decreases by the eigenvalue $\gamma_{geom}$ at each round-trip with:

$$\gamma_{geom,1D} = \frac{1}{\sqrt{|M|}}. \quad (13.11)$$

The power losses per round-trip are then given by:

$$\Pi_{geom,1D} = 1 - |\gamma_{geom,1D}|^2 = 1 - \frac{1}{|M|}. \quad (13.12)$$
Similarly, for a two-transverse-dimension system, we have:

\[
\gamma_{\text{geom},2D} = \frac{1}{|M|},
\]

leading to:

\[
\Pi_{\text{geom},2D} = 1 - |\gamma_{\text{geom},2D}|^2 = 1 - \frac{1}{|M|^2}.
\]

This equation illustrates one of the main advantages of unstable resonators: by choosing the magnification \( M \) of the cavity and the transverse size of the mirrors we are able to make the transverse dimension of the beam fit the diameter of the active medium while maintaining a good transverse mode selection, thus optimizing the energy extraction while keeping a good beam quality. Moreover, the control of the magnification and of the mirror diameter permits to control almost independently the size of the mode and the output coupling. This is why unstable resonators are often preferred to stable resonators when one wants to extract a lot of energy out of a transversally extended amplifying medium. Moreover, since power extraction occurs through the side of the mirror (see figure 13.3) one can use totally reflecting optics. Finally, by using confocal resonators like the ones of figure 13.3, the output beam is automatically collimated.
13.2 Role of diffraction in unstable resonators

As we have just seen (see figure 13.3) the “sizes” of the modes of unstable cavities will be determined by the size(s) of the mirror(s). This implies the fact that the Fresnel diffraction plays a central role in the determination of the modes, contrary to what happens in stable cavities. To introduce this role played by diffraction, we first develop the so-called canonical analysis of unstable resonators.

13.2.1 Canonical formulation

We consider here a 1D resonator (the field is supposed to vary only in the $x$ direction perpendicular to the propagation axis $z$) in which there is only one
mirror (see for example figure 13.3) or one aperture (see for example figure 13.4) that leads to diffraction and output coupling. Starting from a reference plane $z_0$ located just inside this output mirror or just before this coupling aperture (like in figure 13.4), the field in the same reference plane after one round-trip inside the cavity is given, following equations (12.31) and (12.32), by:

$$U_2(x_2) = \sqrt{-i} \frac{\lambda_0}{B} \int_{-a}^{a} \exp \left[ i \frac{\pi}{B \lambda_0} (Ax_0^2 - 2x_0x_2 + Dx_2^2) \right] U_0(x_0) \, dx_0 ,$$  \hspace{1cm} (13.15)

where $2a$ is the width of the mirror along the $x$ direction and the matrix $ABCD$ describes one round-trip inside the cavity starting from the reference plane.

To put equation (13.15) into its general canonical form, we write the input and output waves in the following form:

$$U_0(x_0) = W_0(x_0) \exp \left[ -i \frac{\pi (A - M)}{B \lambda_0} x_0^2 \right] ,$$  \hspace{1cm} (13.16)

$$U_2(x_2) = W_2(x_2) \exp \left[ i \frac{\pi (D - 1/M)}{B \lambda_0} x_2^2 \right] .$$  \hspace{1cm} (13.17)

According to equation (13.3), these transformations permit to extract the wavefront curvature out of the wavefronts in order to collimate them. Then equation (13.15) becomes:

$$W_2(x_2) = \sqrt{-i} \frac{\lambda_0}{B} \int_{-a}^{a} \exp \left[ i \frac{\pi}{B \lambda_0} (Mx_0^2 - 2x_0x_2 + x_2^2/M) \right] W_0(x_1/M) \, dx_0 \, .$$  \hspace{1cm} (13.18)

This form of the Huygens-Fresnel integral corresponds to propagation through a zero-length telescope of magnification $M$ followed by a free-space propagation through a length $MB$ (see figure 13.5):

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} M & B \\ 0 & 1/M \end{pmatrix} = \begin{pmatrix} 1 & MB \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} M & 0 \\ 0 & 1/M \end{pmatrix} ,$$  \hspace{1cm} (13.19)

We thus see that with this transformation, our cavity is equivalent to a free space propagation through a distance $MB$ from plane $z_1$ located just after the magnification step (see figure 13.5). Indeed, by making the change of variable $x_1 = Mx_0$ in equation (13.18), we obtain:

$$W_2(x_2) = \sqrt{-i} \frac{1}{\lambda_0 MB} \int_{-Ma}^{Ma} \exp \left[ i \frac{\pi (x_1 - x_2)^2}{MB \lambda_0} \right] \frac{W_0(x_1/M)}{\sqrt{M}} \, dx_1 .$$  \hspace{1cm} (13.20)

This equation means that a free space propagation of length $MB$ is applied to a $M$ times magnified version of the input wave $W_0(x_0)$. 

13.2.2 Collimated Fresnel number

The so-called *collimated Fresnel number* $N_c$ which characterizes this effective free-space propagation is given by:

$$ N_c = \frac{(Ma)^2}{MB\lambda_0} = \frac{Ma^2}{B\lambda_0}. \quad (13.21) $$

As usually in Fresnel diffraction, this number characterizes the number of diffraction ripples that can be expected in the output beam.

13.2.3 The demagnifying solution

As we have just seen, the magnifying geometrical eigensolution (see section 13.1) in the canonical formulation is a collimated plane wave which i) passes through the aperture, ii) is transversely magnified by a factor $M$, and iii) propagates through distance $MB$ before starting another round-trip. This evolution is represented in figure 13.6(a). As we have seen in section 13.1, the geometric optics approach also predicts the existence of a demagnifying geometrical solution which is in general not collimated, as shown in figure 13.6(b). Let us determine the characteristics of this solution.

If the demagnifying wave has a wavefront radius of curvature $R_0$ in the $z_0$ reference plane, then its transverse stretching by a factor $M$ will multiply this radius of curvature by $M^2$. Indeed, since the phase lag at the outer edge of the beam must remain identical before and after magnification, and since this phase lag evolves like $\exp[ikx^2/2R]$, a magnification of $x$ by a factor $M$ leads to a multiplication of $R$ by $M^2$. We thus have, in the $z_1$ reference plane:

$$ R_1 = M^2R_0. \quad (13.22) $$
This spherical wave then propagates through the free space distance $MB$, leading in the reference plane $z_2$ to a spherical wave with a wavefront radius of curvature $R_2$ given by:

$$R_2 = R_1 + MB = M^2 R_0 + MB \quad .$$

For this wave to be a geometrical eigensolution of our cavity, we must have $R_2 = R_0$, leading to:

$$R_0 = -\frac{MB}{M^2 - 1} \quad .$$

### 13.2.4 Equivalent Fresnel number

Now we have determined the magnifying and demagnifying geometrical solutions of our canonical system, which are strictly valid only for an unbounded system, we can imagine the role played by the diffraction by the finite aperture of width $2a$. Actually, at each round-trip, a part of the magnifying eigenwave will be diffracted by the aperture and partly coupled into the demagnifying eigenwave. This demagnifying wave will experience a few round-trips inside the cavity without being significantly diffracted by the aperture, before being focused strongly enough to eventually experience diffraction again and start spreading to be coupled into the magnifying wave again.

From this qualitative description, one can thus expect the actual behavior of the unstable cavity mode and, *inter alia*, of its losses, to depend on the
relative phase with which the demagnifying wave is excited by the diffraction of the magnifying wave and then injected again into the magnifying wave. Since the magnifying wave is planar and the demagnifying wave has a spherical wavefront with radius of curvature $R_0$ in the aperture plane (see figure 13.6), this relative phase shift is given by:

$$\phi_{\text{mag}}(x = a) - \phi_{\text{demag}}(x = a) = \frac{\pi a^2}{R_0 \lambda_0}. \quad (13.25)$$

We thus expect the behavior of the resonator mode to exhibit some kind of periodicity related to the value of the phase shift of equation (13.25) modulo $2\pi$. We thus introduce the so-called equivalent Fresnel number:

$$N_{\text{eq}} = \frac{|\phi_{\text{mag}}(x = a) - \phi_{\text{demag}}(x = a)|}{2\pi}, \quad (13.26)$$

which reads, using equations (13.21), (13.24), and (13.25):

$$N_{\text{eq}} = \frac{a^2}{2R_0 \lambda_0} = \frac{M^2 - 1}{2M} - \frac{D - M}{B} N_c. \quad (13.27)$$

Now that we know the physical interpretation of $N_{\text{eq}}$, its expression can be directly derived without the use of the canonical approach and using equation (13.3), namely:

$$2\pi N_{\text{eq}} = \phi_{\text{mag}}(x = a) - \phi_{\text{demag}}(x = a)$$

$$\quad = \left( \frac{1}{R_0} - \frac{1}{R_a} \right) \frac{\pi a^2}{\lambda_0}$$

$$\quad = \left( \frac{D - 1/M}{B} - \frac{D - M}{B} \right) \frac{\pi a^2}{\lambda_0}, \quad (13.28)$$

leading to the same result as equation (13.27).

### 13.2.5 Mode losses

Figure 13.7 represents the result of a Fox and Li calculation of the losses of the fundamental mode of a symmetric strip cavity characterized by $g = 1 - L/R$ versus its Fresnel number $N = \frac{a^2}{L \lambda_0}$. One can see that as soon as the cavity becomes unstable ($|g| > 1$), the losses oscillate versus $N$, contrary to the case of the stable cavity in which the losses decrease monotonically versus $N$.

If now the same calculation is plotted versus the equivalent Fresnel number $N_{\text{eq}}$, we can see that the losses exhibit a periodicity related to $N_{\text{eq}}$, and
Figure 13.7: (a) Fox and Li calculation of the losses of the fundamental mode of a symmetric strip two-mirror resonator versus Fresnel number $a^2/L\lambda_0$.
(b) Same results plotted versus the equivalent Fresnel number $N_{eq}$. The horizontal dashed lines correspond to the geometric losses (Taken from A. E. Siegman, Lasers, op. cit.).
oscillate around the value given by the geometrical approximation in equation (13.12). Moreover, we can see that the maxima of the losses correspond actually to mode crossings. Thus, contrary to what happens in stable resonators, the mode exhibiting the lowest losses is not always the same, making the labeling of modes difficult in unstable cavities.

Another example is given in figure 13.8 for the case of a circular resonator. By assuming a shape of the field proportional to \( \exp (il\theta) \) with \( l \) integer and where \( \theta \) is the polar angle in cylindrical coordinates, one can obtain solutions for different values of the azimuthal number \( l \). Figure 13.8 reproduces two examples for \( l = 0 \) and \( l = 1 \). Once again, one can see that, for the different modes, the losses oscillate periodically with \( N_{eq} \) around their geometrical value, thus leading to mode crossings.

![Figure 13.8](image)

Figure 13.8: Numerical calculations of the modulus of the eigenvalue \( \gamma \) of the \( l = 0 \) (upper plot) and \( l = 1 \) (lower plot) azimuthally varying eigenmodes of a circular resonator with magnification \( M = 2 \) versus equivalent Fresnel number (Taken from A. E. Siegman, Lasers, op. cit.).